DISTRIBUTED LEARNING IN LARGE POPULATIONS

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SUMMARY

Distributed learning is the iterative process of decision-making in the presence of other decision-makers. In recent years, researchers across fields as disparate as engineering, biology, and economics have identified mathematically congruous problem formulations at the intersection of their disciplines. In particular, stochastic processes, game theory, and control theory have been brought to bare on certain very basic and universal questions. What sort of environments are conducive to distributed learning? Are there any generic algorithms offering non-trivial performance guarantees for a large class of models?

The first half of this thesis makes contributions to two particular problems in distributed learning, self-assembly and language. Self-assembly refers to the emergence of high-level structures via the aggregate behavior of simpler building blocks. A number of algorithms have been suggested that are capable of generic self-assembly of graphs. That is, given a description of the objective they produce a policy with a corresponding performance guarantee. These guarantees have been in the form of deterministic convergence results. We introduce the notion of stochastic stability to the self-assembly problem. The stochastically stable states are the configurations the system spends almost all of its time in as a noise parameter is taken to zero. We show that in this framework simple procedures exist that are capable of self-assembly of any tree under stringent locality constraints. Our procedure gives an asymptotically maximum yield of target assemblies while obeying communication and reversibility constraints. We also present a slightly more sophisticated algorithm that guarantees maximum yields for any problem size. The latter algorithm utilizes a somewhat more presumptive notion of agents' internal states. While it is unknown whether an algorithm providing maximum yields subject to our constraints can depend only on the more parsimonious form of internal state, we are able to show that such an algorithm would not be able to possess a unique completing rule— a useful feature for analysis.

We then turn our attention to the problem of distributed learning of communication protocols, or, language. Recent results for signaling game models establish the non-negligible possibility of convergence, under distributed learning, to states of unbounded efficiency loss. We provide a tight lower bound on efficiency and discuss its implications. Moreover, motivated by the empirical phenomenon of linguistic drift, we study the signaling game under stochastic evolutionary dynamics. We again make use of stochastic stability analysis and show that the long-run distribution of states has support limited to the efficient communication systems. We find that this behavior is insensitive to the particular choice of evolutionary dynamic, a fact that is intuitively captured by the game's potential function corresponding to average fitness. Consequently, the model supports conclusions similar to those found in the literature on language competition. That is, we expect monomorphic language states to eventually predominate. Homophily has been identified as a feature that potentially stabilizes diverse linguistic communities. We find that incorporating homophily in our stochastic model gives mixed results. While the monomorphic prediction holds in the small noise limit, diversity can persist at higher noise levels or as a metastable phenomenon.

The contributions of the second half of this thesis relate to more basic issues in distributed learning. In particular, we provide new results on the problem of distributed convergence to Nash equilibrium in finite games. A recently proposed class of games known as stable games have the attractive property of admitting global convergence to equilibria under many learning dynamics. We show that stable games can be formulated as passive input-output systems. This observation enables us to identify passivity of a learning dynamic as a sufficient condition for global convergence in stable games. Notably, dynamics satisfying our condition need not exhibit positive correlation between the payoffs and their directions of motion. We show that our condition is satisfied by the dynamics known to exhibit global convergence in stable games. We give a decision-theoretic interpretation for passive learning dynamics that mirrors the interpretation of stable games as strategic

environments exhibiting self-defeating externalities. Moreover, we exploit the flexibility of the passivity condition to study the impact of applying various forecasting heuristics to the payoffs used in the learning process. Finally, we show how passivity can be used to identify strategic tendencies of the players that allow for convergence in the presence of information lags of arbitrary duration in some games.

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This thesis is dedicated to the memory of Avi Pincus.

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CHAPTER 1

INTRODUCTION

In recent decades engineered systems have begun to approach the scale and complexity of biological and social systems. These advances have sparked significant research interest in modeling, prediction, and control of large scale interconnected systems. These problems lie at the intersection of engineering, social, and natural sciences. General problem formulations in this domain suffer from typical tractability limitations such as high dimensionality, constraints on communication and information processing, and fragility due to unmodeled phenomena. Despite these pitfalls, we observe biological systems, transportation, networks, and the internet, among others, managing to function efficiently without substantial centralized infrastructure for aggregating local information and coordinating decisions. Evidently these systems possess a high level of structure. It is this structure that enables these systems to behave efficiently.

This thesis identifies structural properties of large-population multi-agent systems.These structures provide theoretical guarantees for performance and allow for tractable analyses that facilitate predictions.

In the first half of this thesis we study two particular examples of large-scale multi-agent systems arising in engineering and natural science. These are self-assembly and language evolution. First, we consider the problem of programmable self-assembly. Programmable self-assembly is an engineering problem concerned with identifying local interaction rules that enable an ensemble of distributed agents to aggregate into a desired network configuration. These agents could be mechanical, biological, or even inorganic chemical depending on the particular application. While there is a long history of interest in such systems, theoretical possibility results for general, abstract models have only begun to appear in recent years. We provide a constructive algorithm that demonstrates the possibility of self-assembly of graphs under communication constraints. Existing procedures [1],

[2] required communication between connected agents in order to achieve self-assembly. Our algorithms give an asymptotically optimal performance guarantee without the need for communication. In general, if one can assume less about agents' communication and computation capabilities in a model then any conclusions reached will be applicable to a broader range of real application domains.

The form of convergence observed in our self-assembly model is stochastic stability. Loosely, the stochastically stable states are the states observed with non-vanishing probability in the long run for sufficiently small levels of noise. Stochastic stability will be an important overarching theme for the dissertation. This is because stochastic stability is a relatively weak form of probabilistic convergence compared to stronger notions like convergence almost surely. In large-population multi-agent systems there are often elements of persistent idiosyncratic randomness that negate the possibility of stronger forms of convergence. Stochastic stability is also often easy to establish. In fact, it has been used to describe performance guarantees for many algorithms in the context of potential games. The potential game property is a very general and useful structural feature that we will have a lot to say about. In fact, generalizing potential games is taken up later on in this thesis.

We provide a second algorithm that gives optimal performance for any problem size, as opposed to only in the asymptotic sense. This improvement comes at a price, however. While we continue to obey stringent communication constraints, we require a more presumptive notion of state for the agents. Assuming that agents have access to some on-board dynamic memory may be reasonable in many contexts, but there are certainly applications where this is problematic (e.g. chemicals). We provide some theoretical basis for the conjecture that no algorithm can guarantee optimal performance without such a notion of state.

The second problem studied in the first half of this thesis is concerned with mathematical modelling of language learning, particularly as it pertains to early language evolution. While at face-value a science problem, it is also of interest to engineers because understanding conditions and procedures conducive to language acquisition has important consequences for artificially intelligent systems. Our work is motivated by surprising recent results in the literature showing that, in a particular model, evolution may not lead to the emergence of an efficient communication system [3], [4], [5]. The result is troubling precisely because the underlying model is indeed very sensible. The model is of the game-theoretic variety. We will concentrate on game-theoretic models for the remainder of the dissertation [1]. We show that despite potentially unbounded efficiency losses, the emergence of a binary communication system can be guaranteed for almost all initial conditions.

Different approaches exist for explaining away the inefficiency exhibited by the language model. One unrealistic feature of the model is the assumption of an infinite population. This is a powerful heuristic in large-population multi-agent systems. Essentially, using an infinite population (really a continuous mass of agents) can be shown to be approximate a sufficiently large population over short time spans. Evolution is normally associated with long time spans, providing a basis for skepticism regarding the infinite population models. We analyse a finite-population version of the model and show that in the long-run evolution guarantees efficient communication in the sense of stochastic stability. Previous analyses showed that efficient states were robust to isolated stochastic shocks [6], but did not demonstrate that persistent random shocks sufficed to "learn" the efficient states.

We next consider the possibility of multiple efficient communication systems existing side-by-side. That is, the question of why linguistic diversity is observed. We suggest the co-evolution of linguistic community structure and language as an explanatory device. While such an approach had been shown to lead to elaborate network structures in a related

[1] We do not use game theoretic terminology explicitly in our work on self-assembly. Game theoretic concepts are nevertheless implicit. We opt to use different terminology and notation for that problem in order to be consistent with the self-assembly literature and because the prevalent notation of graph-grammars is particularly elegant and well-suited.

model that did not feature persistent random excitation [7], our results differ. In our model we find that non-trivial network structures are unstable in the presence of persistent random shocks. That is, we find that the states exhibiting stochastic stability are just those same states that predominated in our model that had no linguistic community structure. However, these conclusions are in the form of stochastic stability, which may not be the most appropriate analysis. For higher noise levels, simulation results suggest that diverse linguistic communities may be persistent. Alternatively, this diversity may be an effectively stable, or metastable, arrangement due to the process exhibiting an extremely slow convergence rate, or, mixing time.

In the second half of this thesis, we assume a more general and abstract approach to the study of distributed learning. More specifically, we make contributions to the field of learning in games, or, evolutionary game theory. Despite around a half century of efforts, the basic question of how distributed agents can "lean to play" an equilibrium in a game theoretic setting is still largely open. This fact is perhaps unsurprising given the sheer number of problems across so many fields that such results would impact. Indeed, in recent years many researchers have produced negative results that have cast some doubt on the validity of more general formulations of this question. These objections come in two basic forms. First, in many formulations the possibility of equilibrium computation alone, even by centralized methods, has been shown to contradict known hardness results from the computer science literature [8]. Second, if mild constraints on the agents access to information and ability to process it are assumed, then games can be constructed for which no learning dynamic can exhibit stable equilibria [9]. Ongoing research has concentrated on two main directions. First, algorithms with very general performance guarantees have been proposed that are successful at learning equilibria if the mentioned considerations are neglected [10], [11], [11]. Second, classes of games gave been identified that do not include the pathological examples the negative results exploit and therefore can lend themselves to efficient distributed learning procedures. We contribute to the latter approach.

The shining example of a class of games with the mentioned “nice” properties is the potential games [12]. Moreover, a very wide range of learning dynamics have been shown to guarantee distributed convergence to equilibrium in potential games. The most significant drawback of the potential game property is that it is defined by equalities. Given a potential game, there is no guarantee that small perturbations of the model will remain potential games. A number of different generalizations of potential games have been suggested, each attempting to preserve a particular subset of the features [13], [14]. A recently suggested class are the stable games [15]. Stable games encompass the somewhat restricted class of potential games known as concave potential games[2], but also include much more. The most attractive property of the stable game structure is that they exhibit stability under a wide range of system dynamics. In control-theoretic terminology, stable games are plants that are stabilized by controllers satisfying certain very reasonable axioms. Unfortunately, stability results for stable games were developed on a case-by-case basis. There was no sufficient condition for the dynamics that guarantees convergence under stable games. We identify such a sufficient condition. We find that stable games can be formulated as passive input-output systems. It follows that learning dynamics satisfying the passivity condition guarantee convergence to equilibria. We show that many dynamics from the literature are in fact passive.

The list of dynamics that are known to exhibit convergence to equlibria in stable games is long. Still, while these dynamics differ in particulars, they are all static mappings from payoffs to changes in strategy. Such dynamics are often referred to as “myopic” because they are insensitive to past payoffs and do not attempt to forecast where payoffs are going. More sophisticated dynamics can be described as higher order dynamical systems. Examples include fictitious play, regret matching, and various of forms of anticipatory augmentations of myopic dynamics. Also, such policies are implicitly assumed in the equilibrium definitions relevant to dynamic and repeated games, but these approaches are

[2]It may be more appropriate to view stable games as an alternative to potential games, rather than a generalization. Suffice it to say they are two very useful concepts with substantial intersection.

rarely constructive. Very little is understood about the behavior of many higher order learning schemes in broad classes of games. We find that the passivity condition introduced for myopic dynamics can be exploited to demonstrate convergence guarantees for higher order dynamics. We show that in a subclass of the congestion games many dynamic versions of passive learning schemes continue to exhibit global convergence to equilibrium. These dynamic modifications include smoothing and anticipation of payoffs as well as other learning schemes that can be arrived at through cascade interconnection with linear systems.

The final set of contributions addresses the presence of information lags in distributed learning. Such lags are ubiquitous in practice, but rarely addressed in models. We extend the results for passivity of stable games and their dynamics to the infinite-dimensional setting to accommodate time delays in the measurement of payoffs. We show that certain crowd-avoiding, or, contrarian tendencies of the agents which can be described as infinite-dimensional modifications of existing learning dynamics have no consequences for global convergence in stable games. Moreover, these tendencies also provide robustness to time delays in many congestion games.

The last chapter of this thesis identifies some central issues that emerge from consideration of the different aspects of the thesis in tandem and suggests some new directions for research.

This thesis is organized as follows. It is divided into three chapters focusing on the three main areas of self assembly, language, and passivity of stable games. Each of these chapters is intended to be self-contained apart from some general mathematical preliminaries reviewed in the next chapter. Therefore a reader interested in only one of these three areas can pick up the necessary background in the next chapter and then skip to the chapter of interest. Readers with a firm understanding of game theoretic learning can probably just proceed directly to the chapter of interest. While each chapter concludes with a discussion of the results, the final concluding chapter provides a more overarching discussion of issues raised for distributed learning more generally.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1 Game theory

Game theory is the study of interactive decision making. These are strategic situations where the consequences of one's choices depend on the choices of others. While the theory has its roots in mathematics and the social sciences, in recent decades games have been used to model systems of interest to the natural sciences and engineering.

A *game* is a triple, $\mathsf{G} = (\mathsf{P}, \mathsf{A}, \mathsf{U})$, where $\mathsf{P} = \{\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_N\}$ is a set of N players, the collection of sets $\mathsf{A} = \{\mathsf{A}^1, \mathsf{A}^2, ..., \mathsf{A}^N\}$ specifies the actions available to each player, and $\mathsf{U} = \{\mathsf{U}^1, \mathsf{U}^2, ..., \mathsf{U}^N\}$ is the set of player-specific utility functions. The utility functions are maps

$$\mathsf{U}^p : \prod_{j=1}^{N} \mathsf{A}^j \to \mathbb{R},$$

indicating the utility each player i receives given a particular joint action. In this thesis we only consider circumstances where P and each A^i are finite[1]. In order to clarify these ideas we visit the classic example of the prisoner's dilemma.

Example 2.1.1 (The prisoner's dilemma) *In the prisoner's dilemma we have* $\mathsf{P} = \{1, 2\}$, $\mathsf{A} = \{\{C, D\}, \{C, D\}\}$, *and utility functions specified by the bimatrix:*

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$1, 1$

Player 1 is the row player and player 2 is the column player. The first element of each pair in the bimatrix gives the utility for the row player and the second element gives the utility for the column player. The highest average utility, 2, is received when both players play C, or, choose to cooperate. However, if one player cooperates, the other player can

[1] We will demonstrate how to model infinite populations in this framework below.

play D, or, choose to defect and improve her utility from 3 to 4. If both players defect, they each receive a utility of 1. §

An immediate question is whether any particular joint action is "the right one"— the actions we would expect to see in practice. The answer to this question is extremely sensitive to considerations that are typically exogenous to the description of the game itself. For example, whether the players are people, robots or animals and any constraints on available information materially impact what sort of play should be expected. Many different solution concepts exist, however most are either generalizations or refinements of the Nash equilibrium. A *pure Nash equilibrium* is a joint action $\mathbf{a}^* = (a^{*,1}, a^{*,2}, ..., a^{*,N}) \in \prod_{j=1}^{N} \mathsf{A}^j \equiv \mathcal{A}$ such that for all $p \in \mathsf{P}$

$$U^p(\mathbf{a}^*) = \max_{a \in \mathsf{A}^p} U^p(a, \mathbf{a}^*_{-p}),$$

where $\mathbf{a}^{-p} = (a^1, a^2, ..., a^{p-1}, a^{p+1}, ..., a^N)$ are the actions other than a_p in the joint action $\mathbf{a}$. In words, a joint action is a pure Nash equilibrium if no player has incentive to unilaterally deviate from the joint strategy given that the other players play according to the joint action. Games can possess zero, one, or several pure Nash equilibria. The prisoner's dilemma has a single unique pure Nash equilibrium, (D, D). The "pure" qualifier restricts each player to a single, deterministic action choice.

In order to guarantee the existence of a Nash equilibrium, we need to allow randomized, or *mixed*, actions. We refer to a mixed action for player p as an element x^p of the $|\mathsf{A}^p|$-dimensional simplex

$$X^p \equiv \{x^p \in \mathbb{R}^{|\mathsf{A}^p|} : \sum_{j=1}^{|\mathsf{A}^p|} x^p_j = 1\},$$

so that x^p is a probability distribution over the pure actions in A^p. We abuse notation somewhat and write the expected utility for player p associated with a mixed joint action $\mathbf{x}$ as

$$\mathsf{U}^p(\mathbf{x}) = E_{\mathbf{a}}[\mathsf{U}^p(\mathbf{a})] = \sum_{\mathbf{a} \in \mathcal{A}} \mathsf{U}^p(\mathbf{a}) \prod_{k \in \mathsf{P}} x^k_{a_k}.$$

A mixed joint action $\mathbf{x}^* \in \prod_{p\in P} X^p \equiv X$ is a *mixed Nash equilibrium* if for each player p

$$\mathsf{U}^p(\mathbf{x}^*) = \max_{x^p\in X^p} \mathsf{U}^p(x^p, \mathbf{x}^*_{-i}).$$

Every finite game has at least one mixed Nash equilibrium [16].

One interpretation of mixed strategies is that players literally randomize over their available actions. An alternative interpretation is that the elements of P are *populations* and the distribution x^p gives the proportion of population p utilizing each strategy available to that population. In this case, each player is an infinitesimal using a particular pure strategy in A^p. We will refer to this viewpoint as the infinite-population, or continuum agent, model. A *population game*[2] is a triple $\mathsf{G} = (\mathsf{P}, \mathsf{A}, \mathsf{U})$, where $\mathsf{P} = \{\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_N\}$ is a set of N populations, the collection of sets $\mathsf{A} = \{\mathsf{A}^1, \mathsf{A}^2, ..., \mathsf{A}^N\}$ specifies the actions available to each population, and $\mathsf{U} = \{\mathsf{U}^1, \mathsf{U}^2, ..., \mathsf{U}^N\}$ is the set of population-specific utility functions. The utility functions are continuous maps

$$\mathsf{U}^p : X \to \mathbb{R}^{|\mathsf{A}^p|},$$

indicating the utility received by users of each strategy in population p. Population games model large populations. In large populations we expect any single individual's actions to have minimal effect on other agents' utilities. Moreover, agents interact anonymously. A Nash equilibrium of the population game is a joint action $\mathbf{x} \in X$ such that $x_i^p > 0$ implies

$$\mathsf{U}_i^p(\mathbf{x}) = \max_{j\in\mathsf{A}^p} \mathsf{U}_j^p(\mathbf{x}).$$

As mentioned previously, Nash equilibria may not be unique. Furthermore, even when a unique Nash equilibrium exists, such play may not be realized under particular dynamics of strategy change. The study of the dynamics of strategy change is referred to as *learning in games* or *evolutionary game theory*[3]. This branch of game theory is of special interest to engineers as it provides insights into the design of systems with predictable and efficient

[2]We conserve notation by overloading the only slightly different formulation above. Which game we refer to will always be clear from context.

[3]There are a number of excellent monographs on the subject, including [17]

behavior. Equilibrium selection concerns the analysis and design of dynamics of strategy change, or *learning dynamics* that select a particular equilibrium from among many. Later on, we will see something akin to equilibrium selection in the context of self-assembly. Deadlocked states are inefficient equilibria of the system that can hurt the performance of naive algorithms. Our algorithms select equilibria with associated performance guarantees.

In the next section we review a special class of games called potential games. Potential games possess a number of attractive properties, some of which relate directly to learning in games.

2.1.1 Potential games

A potential game [12] is a game [4] endowed with a potential function $\Phi : \mathcal{A} \to \mathbb{R}$ such that for any player p, any joint action $\mathbf{a} \in \mathcal{A}$ and any action $a \in \mathsf{A}^p$ we have

$$\mathsf{U}^p(\mathbf{a}) - \mathsf{U}^p(a, \mathbf{a}^{-p}) = \Phi(\mathbf{a}) - \Phi(a, \mathbf{a}^{-p}).$$

In a potential game, the consequences of **any** individual player's change in strategy are implied by the potential function. While general games need not possess pure Nash equilibria, this is not the case for potential games [12].

Theorem 2.1.1 *Every potential game possesses at least one pure Nash equilibria.*

Beyond merely possessing pure Nash equilibria, potential games also provide a straightforward mechanism for agents to learn to play a pure Nash equilibrium, best response dynamics. The best response dynamic is the simplest of the learning dynamics we will consider. At each discrete time instant t, a player p is selected uniformly at random. That player selects a new action that maximizes her utility, with the other players holding their actions fixed, resulting in a new joint action $\mathbf{a}[t+1] = (a[t+1], \mathbf{a}^{-p}[t])$ where

$$a[t+1] = \arg\max_{a \in \mathsf{A}^p} \mathsf{U}^p(a, \mathbf{a}^{-p}[t]).$$

[4]We provide the definition here for the finite-population version of the potential game here. The analog for population games is postponed until our discussion of stable games in Chapter 5.

Since this procedure corresponds to local maximization of Φ, and local maximizers of Φ are pure Nash equlibria, we have the following [12].

Theorem 2.1.2 *Best response dynamics converge to pure Nash equilibria in potential games.*

Of course, best response dynamics do not guarantee convergence to any particular Nash equilibria. In some situations extremely undesirable Nash equilibria exist. We can “select” a particular equilibrium, namely the global maximum of Φ, if we utilize a more sophisticated learning dynamic known as the logit dynamic. We will see more learning dynamics later on, including logit and learning dynamics for population games. The next sections explore in greater depth a motivating example of the potential games— the congestion games.

2.1.2 Infinite population potential games

We have seen the definition of the potential game in the finite-population setting, the infinite-population version [26] is similar. A population game is a potential game if there exists a C^1 function $\Phi : X \to \mathbb{R}$ such that

$$\nabla\Phi(\mathbf{x}) = \mathrm{proj}_{TX}(F(\mathbf{x})) \quad \forall \mathbf{x} \in X,$$

where proj_{TX} is the projection of the payoff vector onto the subspace

$$TX = \prod_{p \in \mathsf{P}} \{z^p \in \mathbb{R}^{|\mathsf{A}^p|} : \sum_{i \in \mathsf{A}^p} z_i^p = 0\}.$$

We note that the gradient operator here cannot be computed in the usual manner because f is a function on the set X having dimension $n - 1$, and thus its partial derivatives do not exist. The appropriate workaround is the affine calculus, see [27]. Potential games have important equilibrium and dynamical properties. The congestion games are an important and well-studied class of infinite population potential games. We do not consider the finite population version here.

2.1.3 Congestion games

Congestion games [18] are models of distributed network resource allocation. In this section we will review some of the basic properties of congestion games. The second half of this thesis includes some new results for this class of games. We consider the infinite-population version of the congestion game, sometimes referred to as the non-atomic congestion game. A congestion game models utilization of a set Γ of *facilities*. Action $a \in \mathsf{A}^p$ utilizes some set $\Gamma_a^p \subset \Gamma$ of facilities. The set $\rho^p(\gamma) = \{a \in \mathsf{A}^p : \gamma \in \Gamma_a^p\}$ indicates actions in A^p that use $\gamma \in \Gamma$. The utilization level of facility γ under joint action distribution $\mathbf{x}$ is

$$u_\gamma(\mathbf{x}) = \sum_{p \in \mathsf{P}} \sum_{a \in \rho^p(\gamma)} x_a^p.$$

Each facility has an non-decreasing cost function $c_\gamma : \mathbb{R}_+ \to \mathbb{R}$. The utilities of a-users in p are

$$\mathsf{U}_a^p(\mathbf{x}) = -\sum_{\gamma \in \Gamma_a^p} c_\gamma(u_\gamma(\mathbf{x})).$$

The metric of social welfare we study is the average utility

$$W(\mathbf{x}) = -\sum_{\gamma \in \Gamma} u_\gamma(\mathbf{x}) c_\gamma(u_\gamma(\mathbf{x})).$$

As mentioned previously, congestion games are potential games. The potential function is

$$\Phi(\mathbf{x}) = -\sum_{\gamma \in \Gamma} \int_0^{u_\gamma(\mathbf{x})} c_\gamma(z) dz.$$

One problem that has been studied extensively in the congestion game literature is efficiency of equilibria. Consider an example:

Example 2.1.2 (A simple two resource game) *Consider a single population with two actions, each utilizing one of two facilities in $\Gamma = \{1, 2\}$. The cost functions are $c_1(u) = u$ and $c_2(u) = 1$. One Wardrop equilibrium $\boldsymbol{x}^* = (1, 0)$ [5] has all users using the first facility, giving a social welfare of one. The optimal allocation is the one maximizing $W(\boldsymbol{x})$, namely $\hat{\boldsymbol{x}} = (\frac{1}{2}, \frac{1}{2})$, which gives $W(\hat{\boldsymbol{x}}) = -\frac{3}{4}$. The social welfare associated with the equilibrium is $W(\boldsymbol{x}^*) = -1$, which is indeed the worst case equilibrium social welfare.*§

[5] We refer to equilibria in the congestion games as Wardrop equilibria due to historical considerations. They are the same as the Nash equilibria.

2.1.4 The price of anarchy

The worst case efficiency loss at equilibrium in games with negative utilities is quantified by the *price of anarchy* [19]

$$\mathcal{P} = \frac{\min_{x^* \in E} W(x^*)}{\max_{\hat{x} \in X} W(\hat{x})},$$

where E is the set of equilibrium joint action distributions. If, as in the language game, the utilities are positive, we define the price of anarchy as

$$\mathcal{P} = \frac{\max_{\hat{x} \in X} W(\hat{x})}{\min_{x^* \in E} W(x^*)}.$$

In words, the price of anarchy is the worst case ratio of optimal social welfare to equilibrium social welfare. We also refer to the price of anarchy of a class of games as the maximum of this ratio among all games in the class. When we introduce a model of language later on, we will find that equilibrium action distributions with constant social welfare can be constructed for any instance of the game. Since the optimal social welfare is unbounded, the price of anarchy for this class of games is unbounded.

The situation for congestion games is not nearly as bleak. If we fix the form of the cost functions then the price of anarchy will be a constant [20].

Theorem 2.1.3 *The price of anarchy for congestion games with non-decreasing affine cost functions is* $\frac{4}{3}$.

In other words, allowing users to choose facilities selfishly results in at most a 33% loss in efficiency relative to the global optimum. However, even this modest optimality gap can be bridged by instituting tolls on the facilities in a particular manner— the so-called Pigouvian tolls [21].

2.1.5 Marginal cost pricing

The reason that equilibrium allocations fail to achieve maximum social welfare is that users do not experience disutility from negative externalities imposed on others. The idea of marginal cost pricing is to institute tolls on the facilities so as to internalize these externalities. The net cost of a particular facility becomes proportional to the marginal contribution to social welfare, so that the individual and centralized problems optimize proportional objectives.

Suppose the non-decreasing, non-negative cost functions each have the form $c_\gamma(u) = a_\gamma u + b_\gamma$. Let $\hat{x}$ be an optimal action distribution and define the toll for each facility $\gamma \in \Gamma$ as

$$\tau_\gamma = a_\gamma u_\gamma(\hat{x}).$$

The equilibria of the tolled game with cost functions

$$\hat{c}_\gamma(u) = c_\gamma(u) + \tau_\gamma$$

maximize social welfare in the original game. The presumption is that the tolls can be redistributed so that they do not impact social welfare other than through incentivizing particular behaviors. Tolling is generally not helpful when the tolls cannot be redistributed [22].

A critical assumption of marginal cost pricing is that users trade-off tolls and congestion in the same units. This assumption is removed by considering cost functions for the tolled game having the form,

$$\hat{c}_\gamma(u) = c_\gamma(u) + \beta\tau_\gamma.$$

This way users equate β toll units with one unit of congestion. A more general model, where different populations trade-off the two quantities differently has been considered. It is shown that algorithms exist for computing tolls that can enforce not only optimal congestions, but **any** feasible congestion [23], [24]. These procedure do however assume that the network manager has full knowledge of the users' preferences. Quantifying the cost of not knowing this information for particular tolling policies is an open issue in congestion pricing. We have constructed an example to illustrate this point.

Example 2.1.3 (Unintended consequences) *Consider again a single population with two actions, each utilizing one of two facilities in* $\Gamma = \{1, 2\}$. *The cost functions are* $c_1(u) = \frac{3}{4}u$ *and* $c_2(u) = 1$. *The optimal action distribution is* $\hat{x} = (\frac{2}{3}, \frac{1}{3})$, *giving a social cost of* $\frac{2}{3}$. *The equilibrium distribution is* $x^* = (1, 0)$, *which has a social cost of* $\frac{3}{4}$. *In order to recapture this efficiency loss the network manager institutes tolls according to marginal cost pricing. The first facility is tolled at a rate* $\tau_1 = \frac{3}{4}\frac{2}{3} = \frac{1}{2}$, *while the second facility is not tolled because* $a_2 = 0$. *Suppose that* $\beta = 2$ *so that users are more sensitive to tolls than congestion. The equilibrium allocation is now* $\tilde{x} = (0, 1)$, *giving a social cost of 1. In effect, the managers' heavy-handed tolling has produced a situation where the social cost is higher than if she had not tolled at all!* §

Networks can be constructed that give arbitrary efficiency losses for large β. However, it is unknown how exactly these worst-case efficiency losses change with β. In the next section we will introduce the concept of stochastic stability, which will be used extensively in the first half of this thesis.

2.2 Stochastic stability

Let $\mathcal{M}^\epsilon$ be an irreducible and aperiodic Markov chain transition matrix over a finite set of states Z for each $\epsilon \in (0, \bar{\epsilon}]$. If for each $z, z' \in Z$ we have

$$\lim_{\epsilon \to 0} \mathcal{M}^\epsilon_{z,z'} = \mathcal{M}^0_{z,z'},$$

for some Markov chain $\mathcal{M}^0$ over Z, and

$$0 < \lim_{\epsilon \to 0} \frac{\mathcal{M}^\epsilon_{z,z'}}{\epsilon^{r(z,z')}} < \infty,$$

for some $r(z, z') \geq 0$ whenever $\mathcal{M}^\epsilon_{z,z'} > 0$ for some $\epsilon > 0$ then $\mathcal{M}^\epsilon$ is a *regular perturbed Markov process*. We call $\mathcal{M}^0$ the unperturbed process. If $\mathcal{M}^\epsilon_{z,z'} = 0$ for all ϵ, then we define $r(z, z') = \infty$. It is straightforward to see that $\mathcal{P}^\epsilon_{m,n}$ is a regular perturbed Markov process, with $\mathcal{P}^0_{m,n}$ being the reducible Markov chain obtained by substituing $\epsilon = 0$.

Let $\mu(\mathcal{M}^\epsilon)$ be the unique stationary distribution associated with $\mathcal{M}^\epsilon$, a state $z \in Z$ is *stochastically stable* if

$$\lim_{\epsilon \to 0} \mu_z(\mathcal{M}^\epsilon) > 0.$$

In order to characterize the stochastically stable states of $\mathcal{P}^\epsilon_{m,n}$, we will make use of the theory of resistance trees [14]. Let $R_1, ..., R_J \subset Z$ be the recurrent communication classes of $\mathcal{M}^0$. Given two recurrent communication classes R_i and R_j, let $\{z_0, z_1, ..., z_K\}$ be a path satisfying $z_0 \in R_i$ and $Z_K \in R_j$. We call the quantity $\sum_{k=0}^{K-1} r(z_k, z_{k+1})$ the *resistance* of the path. With slight abuse of notation we define r_{ij} to be the *least resistance* among all such paths.

Consider a graph G whose vertex set is the set of recurrent communication classes. An R_i-tree T is a spanning tree in G such that for any vertex R_j, $j \neq i$ there is a unique directed path from R_j to R_i. We define

$$\gamma(R_i) = \min_{T \in \mathcal{T}_{R_i}} \sum_{(R_j, R_k) \in T} r_{jk},$$

where $\mathcal{T}_{R_i}$ is the set of all R_i trees in G, which we refer to as the *stochastic potential* of R_i. We have the following theorem [14], which characertizes exactly the set of stochastically stable states.

Theorem 2.2.1 *Let $\mathcal{M}^\epsilon$ be a regular perturbed Markov process and let $R_1, .., R_J$ be the recurrent communication classes of the unperturbed process $\mathcal{M}^0$. Then the stochastically stable states are precisely those states contained in the recurrent communication classes with minimum stochastic potential.*

Example 2.2.1 (A three-state process) *Figure 1 illustrates a simple three state Markov process, P^0. Each state S_i, $i \in \{1, 2, 3\}$ is a singleton recurrent class, also referred to as an absorbing state. The state transition matrix is given by*

$$P^0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

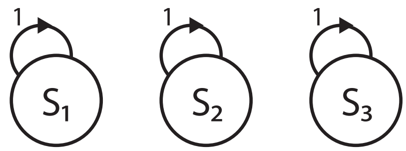

Figure 1: A simple three-state Markov process, P^0. The system stays in its initial state for all t.

It is easy to see that all state distributions are stationary in this example because the transition matrix is the identity matrix. This process is not particularly interesting, until we introduce the perturbation and arrive at P^ϵ as shown in Figure 2. It can easily be verified that P^ϵ is a regular perturbed Markov process, so it must have at least one stochastically stable state. In this example, we can easily see that the minimum resistance rooted trees for each state (as each is a recurrent class of P^0) must be

$$S_1 \to S_3 \to S_2, r = 2 + 1 = 3$$

$$S_3 \to S_2 \to S_1, r = 1 + 1 = 2$$

$$S_2 \to S_1 \to S_3, r = 1 + 2 = 3$$

where the resistance is also noted. All other rooted trees will involve crossing states more than once, so that all other trees rooted at S_i will have a single edge equal to the resistance above in addition to other non-zero resistance edges. We can conclude in this example that there is one stochastically stable state, S_1. This result is intuitively satisfying because among the three states, the probability of leaving S_1, ϵ^2, is least. §

In cases with more edges in P^ϵ and more states, finding the minimum resistance tree rooted at S_i will be more difficult, although it is always a finite search. We do note, however, that the search applies only to recurrent classes of states and these are oftentimes much fewer in number than the states themselves.

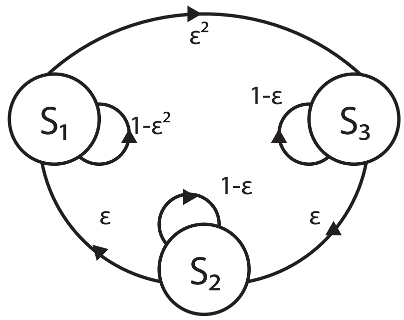

Figure 2: The perturbed process P^{ϵ} has $r(S_1, S_2) = 2, r(S_3, S_2) = 1, r(S_2, S_1) = 1$.

2.2.1 Logit dynamics

In finite population potential games we saw that the best response dynamics converge to a pure Nash equilibrium. However, which equilibrium is reached depends on the starting action profile and the order that players revise their strategies. Other dynamics, notable the logit dynamics, can guarantee that play lingers around a particular subset of equilibria—the global maximizers of the potential function. We review the form of the logit dynamic and its fundamental guarantee for performance. Suppose that at each discrete time instant t, a single player is selected at random uniformly to update her action, with all other players maintaining their current action. The updating player P_i updates her action according to the distribution $p_i(t) \in \Delta(\mathsf{A}^i)$ where

$$p_i^a(t) = \frac{e^{\frac{1}{\epsilon}\mathsf{U}^i(a,\mathbf{a}_{-i}(t-1))}}{\sum_{\bar{a}\in\mathsf{A}^i} e^{\frac{1}{\epsilon}\mathsf{U}^i(\bar{a},\mathbf{a}_{-i}(t-1))}},$$

for each $a \in \mathsf{A}^i$ and the temperature $\epsilon > 0$. The temperature parameter specifies the likelihood of player i updating to a suboptimal action. In the small ϵ limit, we recover the best response dynamics. In a potential game with potential function Φ, the stationary distribution of the joint action $\mathbf{a}$ is [25]

$$\mu(\mathbf{a}) = \frac{e^{\frac{1}{\epsilon}\Phi(\mathbf{a})}}{\sum_{\bar{a}\in\mathfrak{A}} e^{\frac{1}{\epsilon}\Phi(\bar{\mathbf{a}})}}$$

. Thus, for small ϵ the stationary distribution concentrates probability on the states that maximize the potential function. Identifying the parameter ϵ as the perturbation term in a regular perturbed Markov process, it is also the case the the set of potential function maximizers are the stochastically stable states. Stochastic stability is rarely established in this manner. The logit dynamic is one of the rare instances where an explicit characterization of the stationary distribution in closed form exists for all ϵ. In general, tools that characterize only the set of stochastically stable states, such as resistance trees, are utilized. The next section considers dynamics in the infinite population setting.

2.3 Convergence of evolutionary dynamics

Population games model strategic interactions in large populations. Evolutionary dynamics model the behavior of the players. There are three defining features of evolutionary dynamics. First, players only occasionally update their actions. Second, players are short-sighted and thus choose actions in myopic fashion. Third, players possess limited information about other players' strategies. These principles reinforce each other and are motivated by large population settings where gathering information and exact optimization are costly.

Evolutionary dynamics can be shown to closely approximate stochastic decision policies over finite time spans with sufficiently large finite populations [28]. We proceed to formally define evolutionary dynamics.

An *evolutionary dynamic* is a map[6]

$$V : X \to TX.$$

We use the subscript to identify the utility function, i.e. V_{U}. This is because the dynamic is specific to the game, but only through the utility functions. In language, we will encounter the replicator dynamics

$$\dot{x}_i^p = x_i^p[\mathsf{U}_i^p(x) - \sum_{j \in \mathsf{A}^p} x_j^p \mathsf{U}_j^p(x)] = (V_{\mathsf{U}}^p)_i.$$

The general form of the overall system is $\dot{\mathbf{x}} = V_{\mathsf{U}}(\mathbf{x})$.

[6] Although evolutionary dynamics can, in principle, be set-valued maps leading to differential inclusions, we simplify here to keep the presentation concise.

Part of the allure of potential games is that the potential function serves as a Lyapunov function for many dynamics.

Theorem 2.3.1 *Let* $\mathsf{G} = (\mathsf{P}, \mathsf{A}, \mathsf{U})$ *be a potential game with potential function* Φ*. Suppose the evolutionary dynamic* $\dot{\boldsymbol{x}} = V_{\mathsf{U}}(\boldsymbol{x})$ *satisfies*

$$V^p_{\mathsf{U}}(\boldsymbol{x}) \neq \boldsymbol{0} \Rightarrow V^p_{\mathsf{U}}(\boldsymbol{x})' \mathsf{U}^p(\boldsymbol{x}) > 0.$$

Then Φ *is a strict Lyapunov function for* V_{U}*.*

This fact is most of what is needed to show convergence to equilibrium for a broad range of dynamics. The alignment condition in the theorem is known as *positive correlation* (PC). Absent the potential game structure, convergence is far from guaranteed. Indeed, limit cycles and even chaos are produced under certain games. Unfortunately, the potential game is a brittle property. Since it is defined by equalities, perturbations to a potential game will generally lose the potential structure. Since we desire results that are insensitive to modelling details very generally, this is disappointing, to say the least. A recently proposed class of games that is closely related to potential games, stable games, address this issue.

CHAPTER 3

SELF ASSEMBLY

Self-assembly[1] refers to the emergence of high-level structures via the aggregate behavior of simpler building blocks. Researchers have long been interested in understanding self-assembly with an eye towards exploiting this understanding in engineered systems. A number of algorithms have been suggested that are capable of generic self-assembly. That is, given a description of the objective they produce a policy with a corresponding performance guarantee. These guarantees have been in the form of deterministic convergence results. Not unlike in natural self-assembling systems that exhibit and often depend on process noise, we consider the benefits of relaxing our expectations to allow for probabilistic performance guarantees. In particular, we introduce the notion of stochastic stability to the self-assembly problem. The stochastically stable states are the configurations the system spends almost all of its time in as a noise parameter is taken to zero. We show that, in this framework, simple procedures exist that are capable of self-assembly of any tree under stringent locality constraints. Our procedure gives an asymptotically maximum yield of target assemblies while obeying communication and reversibility constraints. We also present a slightly more sophisticated algorithm that guarantees maximum yields for any problem size. The latter algorithm utilizes a somewhat more presumptive notion of agents' internal states. While it is unknown whether an algorithm providing maximum yields subject to our constraints can depend only on the more parsimonious form of internal state, we are able to show that such an algorithm would not be able to possess a unique completing rule— a useful feature for analysis. We examine the impact of the reversibility constraint by showing how each algorithm can be modified to achieve better performance when the constraint is relaxed. We also revisit a related algorithm from the literature to better understand its performance in the context we study. We give analytical proofs of correctness for

[1]The results described in this chapter appear in [29], [30], and [31]

our algorithms and provide simulation results for further insight.

3.1 Introduction

Self-assembly is the phenomenon of an ordered structure emerging from the aggregate behavior of simpler constituent entities acting autonomously. Galaxies self-assemble and so do humans, but little is understood about such processes. Self-assembly has been the subject of a great deal of research. The reasoning is twofold. First, understanding self-assembly generically is essential to elucidating natural self-assembling systems. Second, techniques applicable to the manufacture and operation of self-assembling engineered systems can potentially be developed. The level of complexity inherent in the former circumstance suggests tremendous scalability, reliability, and parallelization advantages for successful exploitation of self-assembly in the latter. While interest in generic self-assembly dates back to at least the 1950's [32], the treatment of foundational possibility results has only begun to appear in the literature in recent years. What follows is intended as a brief overview of the state of the art and an attempt to frame the problems we address in a broader context.

One branch of self-assembly research addresses the issue of origins. That is, how do self-assembling systems emerge spontaneously? The process of natural selection assumes the existence of organisms that self-assemble and subsequently replicate in the presence of selection forces. The question is then begged— how do the blind processes of physics and chemistry conspire to set such processes into motion? Some approaches to this question center around the identification and characterization of intrinsic tendencies towards order in nature. Self-organized criticality refers to a wide range of results spanning across several fields, largely initiated by the seminal paper [33]. Roughly speaking, many dynamical systems (e.g. snowflake formation) with a critical point as an attractor exhibit self-organization. The common theme in this area of self-assembly is the emergence of order from disorder. We do not address this problem. Instead, our agenda is prescriptive. We are interested in inducing self-assembly in models that would not otherwise support it.

The severe assumption in the "programmable" self-assembly processes we explore is that the constituent parts will each need to harbor complexity internally that is increasing in the complexity of the assembly task. For instance, in animal embryos self-assembly can be thought of as an expression of the information in the genome. Therefore, the complexity of the mature animal is already present, although latent, in the earlier stages of development. In contrast with the problem of origins, we are interested not in the production of order from disorder, but the realization of order from an already very structured and explicit instruction set.

Mutation and selection forces are known to modify DNA and augment the complexity and functionality of the organism in the long-run. We are not expressly concerned with this phenomenon either. We are interested in identifying the instructions that can be loaded by the parts and the performance of the induced dynamic systems. We ask, given very general and straightforward dynamics and certain basic constraints on the parts' ability to carry out their instructions, under what circumstances can rules be synthesized such that the system will behave in a predictable, desirable way? In particular, for various constraints on agents capabilities we establish the possibility of generic self-assembly. In each case, we attempt to make the class of achievable assemblies as large as possible while providing the strongest possible performance guarantee. In some cases, the nature of the constraints is such that the performance guarantees must be weaker than in other cases.

The structure of the rule sets and the underlying dynamics that execute the rules will be fixed. We propose algorithms for the synthesis of rules from a description of the assembly goal. Target assemblies are represented by specific graph topologies. We desire that a graph with at least as many nodes as the target graph "assemble" the target graph by having it's nodes create and sever edges according to preloaded local rules. When there are many more nodes than the target, we prefer as many copies of the target to be assembled as possible. Our approach is obviously much more simple than the processes that inspire it. However, we tolerate more stringent constraints than existing generic algorithms in the literature. A key innovation in our approach is our allowing for probabilistic performance guarantees. We utilize the notion of stochastic stability. These are the states in the support of the stationary distribution of a family of Markov chains as a perturbation term is taken to zero.

The ability to achieve self-assembly directives through local rules exclusively is a problem that is relevant to biology, robotics, manufacturing and other application areas. Thus far, self-assembly theory is yet to prove itself a disruptive force in any defining killer app. Consequently, no particular model is especially canonical. The model we introduce will invariably be suited more to some application areas than others. It may not be meaningful to speak of programmability at all in the context of crystallization and polymerization processes that are governed solely by thermodynamics. We introduce constraints on the capabilities of the agents to assuage these concerns. Each application area has its own limits on agents' locomotion, sensing, memory, and computation. We demonstrate that under even very stringent constraints, a high level of self-assembly performance is achievable. Since the possibility of self-assembly is our concern, we do not at this point comment on convergence rates outside of empirical observations based on simulation results.

Our model is straightforward. The system is a graph that evolves over time. Each vertex is an identical atom. The finite number of vertices is fixed at the outset and the set of edges is dynamic. Each node also has an internal state taking on values from a finite set. At each iteration, two nodes are selected at random. If there is a rule in the finite rule set that applies to the nodes, they either apply the rule (changing the graph) or do nothing depending on the probability associated with that particular rule. If multiple rules apply, one is selected at random. The rules are described using the notation of graph grammars [34]. We desire a maximum number of disjoint maximal connected subgraphs isomorphic to a target graph. More plainly, we want to maximize the yield of desirable assemblies.

We operate under communication and reversibility constraints. The communication constraint is observed in all of the algorithms we present, it enforces a strong notion of locality in our procedures because information cannot propagate (e.g. by way of a connected graph) so that ostensibly global information concerning graph structure can be made available to the agents. This way, we are faced with a decision problem at the individual component level as opposed to the sub-assembly level. Reversibility is a necessary property in many application areas— we explore the impact of this constraint by considering both processes that obey and neglect it. Beyond these constraints we pay special attention to the internal states that our algorithms require the agents to maintain. Internal states that can be recovered from the unlabeled graph (up to an isomorphism) are desirable because they imply that heterogeneity in agent behavior is a consequence of their position in a graph only. This way, we do not assume that identical atoms are distinguished from each other based on anything other than their roles in particular graph topologies. To illustrate this distinction we make an analogy to the hydrogen atom. A hydrogen atom participating in hydrogen gas (H2) would not be expect to behave similarly to a lone hydrogen atom. However, we expect every hydrogen gas molecule and every lone hydrogen atom to behave in the same way. Whether it is reasonable to expect the agents to maintain internal states that cannot be recovered from the unlabeled graph is application-dependent.

Another desirable feature of the internal states is uniqueness. Once an assembly is completed it is advantageous for each part to have a different state. This way the process will be naturally composable in higher level self-assembly processes. Rules can be synthesized that use the states of complete assemblies at the lower level as a starting point. If there is redundancy than their will be limitations on the configurations that the higher level process can realize. We suggest a very simple procedure that gives maximum yields asymptotically in the total number of available parts. A slightly more sophisticated procedure gives maximum yields for any number of parts, but introduces internal states that cannot be recovered from the graph. We suspect that algorithms giving both maximum yields and recoverable internal states exist. However, if we insist on uniqueness of states in complete assemblies as well, the situation is less clear. We show that a feature our analysis depends on, the presence of a unique completing rule, can never be guaranteed for an algorithm with both unique and recoverable states.

The outline of this chapter is as follows. In Section 3.2 we highlight other self-assembly research that is either prior or parallel to our own. Section 3.3 provides the definitions we will rely upon. We give a simple algorithm with asymptotically maximum yields in Section 3.4. We introduce an algorithm that always gives a maximum yield in Section 3.5. We compare the two algorithms and comment on potential consequences for the internal states when we insist on maximum yields in Section 3.6. The final two sections of this chapter describe an algorithm combining some desiderata of both our algorithms, and a look at how an existing algorithm from the literature performs in our model.

3.2 Related Work

The synthesis problem for programmable self-assembly of graphs was introduced in [2]. There, the procedure depends upon communication between agents participating in an assembly and decisions are made according to a policy that relies on exhaustive search through all possible sub-assemblies. The notion of deadlock (multiple partial assemblies as undesirable equilibria) is also introduced. In [1] the formalism of graph grammars is first utilized in self-assembly of graphs and algorithms for synthesizing rules are presented. In particular, the `MakeTree` algorithm uses only constructive and destructive binary rules so that our communication constraint is observed. This procedure has a performance guarantee for all acyclic graphs when the number of agents is infinite. However, to avoid deadlock when the number of agents is finite, a disassociation rule must be added which depends upon implementation of a consensus algorithm inconsistent with the communication constraint we insist on. This stream of work has contributed many other results in this area including optimal non-deterministic behavior for some cases, and a robotic programmable parts testbed [35].

Designing self-assembly rules that are optimal with respect to convergence rates subject to a probabilistic performance constraint was considered in [36]. Stochastic stability has also been used as an equilibrium concept in a mildly related network formation game [37]. A similar notion of stability has been applied to the analysis of gene regulatory networks [38].

Another stream of research has represented programmable self-assembly using cellular automata [39]. The generic algorithms are applicable to all assemblies that are filled, non-cantilevered, and convex in each layer. However, the agents are assumed to know their exact global position at all times. This can be guaranteed as long as the agents know their positions initially.

Another model that has actually seen some experimental success is the tile [40] (another form of cellular automata). Basic self-assembly and computation capabilities have been demonstrated with DNA-based tiles. This model also has various associated theoretical results relating to computational and self-assembly tasks, see for instance [41].

Numerous robotic self-assembling systems have been developed, notably [42] and [43]. Some mathematical formalization of these methods has also been done [44]. General global-to-local techniques for self-assembly are considered in [45]. A synopsis of various contributions in robotic self-assembly is available [46].

While most approaches to self-assembly have focused on structural assembly tasks, [47] has instead emphasized the function of resulting assemblies.

3.3 Definitions

3.3.1 Graph Grammars

In this section we succinctly reproduce the notion of graph grammars introduced in [1]. We will use a slightly different formulation that is tailored to our setting. A simple labeled graph is a triple $G = (V, E, l)$ where $V = \{1, ..., N\}$ are vertices (or parts), $E \subset V \times V$ are pairs of vertices (or edges), and $l : V \to \mathcal{S}$ is a labeling function indicating the internal state of each node. The number of identical atoms, or parts, is N. Parts are attached if their indices are one of the pairs in E. Pairs of nodes $\{x, y\} \in E$ are denoted by xy. The label $l(x)$ of a part x is its internal state information from the finite set of states $\mathcal{S}$. We use the subscript notation V_G, E_G, l_G to refer to respectively, the vertex set, edge set, and labeling function of a graph G. We also use $n_E(k)$ to refer to the neighbors of vertex k relative to the edge set E. The set of unlabeled graphs with vertex set V is denoted $\mathcal{G}_V$.

Our self-assembly objectives will be related to E only. V will be static and l will influence how E changes, but will not be material to our objectives intrinsically. In this framework, assemblies are network topologies. In [30] we use weighted graphs to confer geometric orientations on the edges, but we omit these details here in the interest of simplicity.

We say that two graphs are an *isomorphism*, or one graph is *isomorphic* to another when they obey an equivalence relation. That is $G_1 \simeq G_2$ if $\exists h : V_{G_1} \to V_{G_2}$ bijective such that

$$ij \in E_{G_1} \Leftrightarrow h(i)h(j) \in E_{G_2}.$$

The isomorphism is *label-preserving* if $l_{G_1}(x) = l_{G_2}(h(x)) \ \forall \ x \in V_{G_1}$.

Due to the vertices being identical atoms, any element of an equivalence class of graphs represents the same assembly. Since it is self-assembly performance that we are concerned with, our objective will be phrased in terms of equivalence classes of graphs.

Given $I \subset V$ we define the *subgraph* $G \cap I = (V \cap I, E \cap I \times I, l|_{I \times I})$. We say that G contains H if a subgraph of G is isomorphic to H. A connected subgraph is maximal if there are no nodes in the original graph that could have been added to the subgraph while still leaving the subgraph connected. We will use the terms assembly and maximal connected subgraph interchangeably.

Definition 3.3.1 *A* ***rule*** *is an ordered pair of graphs $r = (L, R)$ such that $V_L = V_R$. The graphs L and R are the* ***left hand side*** *and* ***right hand side*** *of r. The* ***size*** *of r is $|V_L| = |V_R|$.*

We refer to rules of size two as *binary* rules. If $E_L \subsetneq E_R$ a rule is called *constructive*. If $E_L \supsetneq E_R$ a rule is called *destructive*. Otherwise, the rule is *mixed*. Note that we define these set inequalities strictly, unlike some others. Visually we can represent a binary rule as

$$a - b \rightharpoonup c \quad d$$

where the letters are the labels and the vertices are suppressed, with the left node of the left hand side corresponding to the left node of the right hand side, and similarly for the right nodes. A rule represents a local change in a graph, i.e. $|V_G| \geq |V_L|$.

Definition 3.3.2 *A rule $r = (L, R)$ is **applicable** to a graph G if there exists a subgraph $G \cap I$ and a label-preserving isomorphism $h : V_G \cap I \to V_L$. In this case h is called a **witness** and the triple (r, I, h) is called an **action**.*

Definition 3.3.3 *When (r, I, h) is an action with $r = (L, R)$ on G, the **application** of (r, I, h) to G gives a new graph $G' = (V_G, E_{G'}, l_{G'})$ defined by*

$$E_{G'} = (E_G - \{xy : xy \in E_G \cap I \times I\}) \cup \{xy : h(x)h(y) \in E_R\}$$

$$l_{G'}(x) = \begin{cases} l_G(x), & \text{if } x \in V_G - I \\ l_R \circ h(x), & \text{otherwise} \end{cases}$$

We write $G \xrightarrow{r,I,h} G'$ to indicate that G' was obtained from G via application of (r, I, h). We define the *complement* of a rule as $\bar{r} = (R, L)$ so that $G \xrightarrow{r,I,h} G' \xrightarrow{\bar{r},I,h} G'' = G$.

If we have a set of rules Φ then we can begin to examine sequences of graphs obtained from successive application of the rules.

Example 3.3.1 (Simple cycle-building rules) *Consider the following set of constructive binary rules:*

$$\Phi = \begin{cases} a \quad a \rightharpoonup b - c, & (r_1) \\ c \quad a \rightharpoonup d - e, & (r_2) \\ e \quad b \rightharpoonup f - g. & (r_3) \end{cases}$$

From the initial graph $G_0 = (\{1, 2, 3\}, \{\emptyset\}, l_0(\cdot) = a)$ there is only one possible trajectory if we insist on applying the unique applicable rule at each step, shown in Figure 3. §

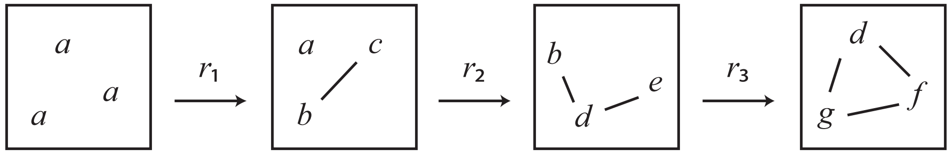

Figure 3: The rules in Example 3.3.1 can be applied successively to generate the cycle on the right. The subgraphs and witnesses should be obvious from the figure.

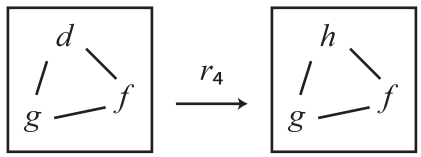

Figure 4: r_4 effectively acts as a communication step, updating the agent labeled d that the cycle has been closed.

Example 3.3.2 (Binary communication) *Continuing with the previous example, consider the label d. When r_3 is applied, the chain closes into a cycle, but the node with label d is unaffected. Considering labels as representing the local information available to each agent, the agents with labels f and g know the exact structure of the graph since these labels are only adopted coinciding with r_3. If we augment the rule set with a mixed rule:*

$$\hat{\Phi} = \Phi \cup \{d - f \;\rightharpoonup\; h - f, \quad (r_4)\}$$

then the agent labeled d is apprised that the cycle is completed by its neighbor with label f, so that in the final graph, all agents are aware of the complete structure of the assembly they participate in. The effect of r_4 is illustrated in Figure 4. §

In order to disallow communication, the algorithms we present will be constrained so that they can only synthesize a finite number of binary rules— each one being either constructive of destructive. We also point out that if the number of vertices in the example where greater, it would be possible for r_3 to occur between two different subgraphs, producing a long chain instead of a cycle. This reflects a very general limitation with finite binary rule sets [1]. For this reason, we will concentrate only on acyclic assembly objectives.

Since we will only be concerned with binary rules, we will introduce random pairwise selection dynamics to place the application of rules in a systematic framework.

3.3.2 Random pairwise selection dynamics

A random pairwise selection dynamic graph is a quadruple $\Sigma = (G_0, F, \Phi, \mathcal{R})$. The graph G_0 is an initial condition. The set Φ is the rules. The family of random variables $F(G), G \in \mathcal{G}_{V_{G_0}}$ take on values in $\{(x, y) : x, y \in V_{G_0}, x \neq y\}$ so that each $F(G)$ selects two vertices without replacement. $\mathcal{R} : \Phi \to (0, 1]$ assigns a Bernoulli distribution parameter to each rule. With these definitions, we can generate a random sequence of graphs $\{G_t\}_{t=1}^{\infty}$ as follows:

1. Initialize with $t = 0$ and G_0.

2. Increment t.

3. $F(G_t)$ is realized, giving a pair of vertices $\{x, y\}$.

4. Let
$$\Phi_t = \{r \in \Phi : \exists\, h \text{ s.t. } (r, \{x, y\}, h) \text{ is an action on } G_{t-1}\}.$$

5. If $\Phi_t = \{\emptyset\}$ let $G_t = G_{t-1}$ and return to step 2.

6. Let $r \in \Phi_t$ be chosen at random, uniformly.

7. Let $G_{t-1} \xrightarrow{r,\{x,y\},h} G'$.

8. Let
$$G_t = \begin{cases} G', & \text{w.p. } \mathcal{R}(r) \\ G_{t-1}. & \text{w.p. } 1 - \mathcal{R}(r) \end{cases}$$

9. Return to step 2.

We will be interested in characterizing the asymptotic behavior of $\{G_t\}$ for various choices of Φ and $\mathcal{R}$. The random sequence of selections, $F(G_t)$ will be considered exogenous. Sampling from $F(G_t)$ gives an inherent stochasticity to the process even if $\mathcal{R}(\cdot) = 1$,

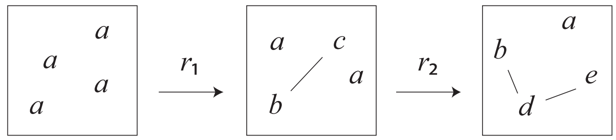

Figure 5: Successful realizations of $\{G_t\}$ occur with positive probability

i.e. no random behavior is introduced intentionally. Random pairwise selection dynamics can therefore be thought of as a model in which agents interact via random encounters and then behave according to the rules and their associated probabilities. The interaction probabilities depend on the current graph G_t. Since $F(G_t)$ is exogenous, we will have limited control over the trajectories of $\{G_t\}$, still, we hope to influence the long-run properties of the system through Φ and $\mathcal{R}$. This model is appropriate for systems where agent motion is stochastic, such as in a liquid solution. Alternatively, we can think of the model as corresponding to a system with deterministic agent motion that is abstracted away or approximated via random encounters.

The selection at time t, $F(G_t)$, could alternatively be defined so that the probabilities associated with various selections depend on a finite history $\{G_\tau\}_{\tau=t-T}^{t}$. Our results will depend on the overall system being finite-state Markov and an assumption that the probability associated with any particular selection is always bounded away from zero. So long as these assumptions are obeyed, the exact statistical dependencies of the selection process F are immaterial to our analyses. More formally, we assume throughout that there exists $\bar{F} > 0$ such that

$$\mathbf{Pr}\left[F(G) = \{i, j\}\right] \geq \bar{F} \quad \forall\, i, j \in V, i \neq j \text{ and all } G \in \mathcal{G}_{V_0}.$$

The specific method used to pick between rules in step 6 could also be more general. Again, at issue is that the probability of selecting any particular agent is bounded away from zero. The reasoning behind this requirement will become clear later on.

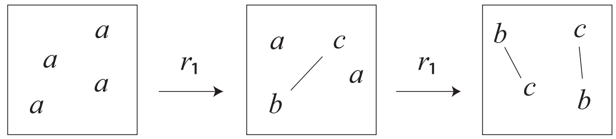

Figure 6: Unfortunately, the system in Example 3.3.3 can exhibit deadlock.

3.3.3 The self-assembly problem

Let G_0 be an initial graph and $\hat{G}$ an unlabeled target graph. The *yield* of a graph G with respect to a target $\hat{G}$, $\mathcal{Y}_{\hat{G}}(G)$, is the number of disjoint maximal connected subgraphs in G that are isomorphic to $\hat{G}$. Building on this definition we define the set:

$$\mathcal{G}^{\hat{G}}_{V_{G_0}} = \{G : V_G = V_{G_0}, \mathcal{Y}_{\hat{G}}(G) = \lfloor |V_{G_0}|/|V_{\hat{G}}| \rfloor\}$$

as the set of maximum yield graphs. For all the graphs in $\mathcal{G}^{\hat{G}}_{V_{G_0}}$, it is impossible for any rules to increase the number of completed assemblies. We do not specify any preference for the remainder nodes when $|V_{G_0}|$ is not an integer multiple of $|V_{\hat{G}}|$.

The self-assembly problem is, given F and G_0, to find a set of rules Φ and associated probabilities $\mathcal{R}$ so that $\{G_t\}$ will enter and remain in $\mathcal{G}^{\hat{G}}_{V_{G_0}}$.

Example 3.3.3 (Deadlock) *Consider the system $\Sigma = (G_0, F, \Phi, \mathcal{R})$ defined by*

$$G_0 = (\{1,2,3,4\}, \{\emptyset\}, l_0(\cdot) = a)$$

$$F \sim \textit{ i.i.d. uniform}$$

$$\Phi = \begin{cases} a \quad a \rightharpoonup b - c, & (r_1) \\ c \quad a \rightharpoonup d - e, & (r_2) \end{cases}$$

$$\mathcal{R}(\cdot) = 1.$$

Suppose $\hat{G} = (\{1,2,3\}, \{12, 23\})$. Figure 5 gives a possible trajectory for $\{G_t\}$. In this case, the process was successful since $G_t \in \mathcal{G}^{\hat{G}}_V$ for all $t \geq 2$. However, another possible trajectory is shown in Figure 6. In this case, the system has reached an undesirable steady state and we have $G_t \notin \mathcal{G}^{\hat{G}}_V$ for all t. Notice that each maximal connected subgraph of G_t

is isomorphic to a subgraph of $\hat{G}$— this is the phenomenon referred to as deadlock [2]. Deadlock is an issue because we consider G_0 with finitely many vertices only, so the supply of parts can become exhausted in undesirable graphs that are invariant under Σ. This issue is addressed in [1] for a very similar situation. It is suggested that the agents run a consensus algorithm to estimate if deadlock has occurred, and if they deem it has, to sever their edges so that they will be available to complete other assemblies. In this paper we will show that deadlock can be avoided without recourse to communication. §

Depending on the constraints introduced on Φ and $\mathcal{R}$ it may not be possible to make $\mathcal{G}_V^{\hat{G}}$ an invariant set of the system Σ. In this case, we will be limited to making probabilistic statements about $\mathcal{Y}_{\hat{G}}(G_t)$. We introduce one of these constraints now.

3.3.4 Reversibility

One very natural constraint on Φ and $\mathcal{R}$ is related to the reversibility of the various rules. In many settings, reversibility is a necessary constraint in order for models to be realistic [48], [49].

Definition 3.3.4 *The pair $(\Phi, \mathcal{R})$ is **reversible** if for any $r \in \Phi$ we have $\overline{r} \in \Phi$.*

Those familiar with chemical reaction networks will recognize this definition as analogous to the notion of reversibility in that context. Later we will analyze $\{G_t\}$ as a Markov process. The reader is advised that the above definition of reversibility does not imply that $\{G_t\}$ is a reversible Markov process. A reversible Markov Process with state transition matrix P satisfies the detailed balance condition

$$P_{ij}\pi_i = P_{ji}\pi_j$$

for all i, j, where π_i and π_j are the stationary probabilities associated with states i and j, respectively. The notion of reversibility in Definition 3.3.4, in terms of the Markov process $\{G_t\}$ (with each possible graph a state), is $P_{ij} > 0 \Leftrightarrow P_{ji} > 0$.

Clearly it is impossible for $\{G_t\}$ to stay in $\mathcal{G}_V^{\hat{G}}$ when $(\Phi, \mathcal{R})$ is reversible. Because of this, the best we can do is synthesize Φ and $\mathcal{R}$ so that $\{G_t\}$ will be in $\mathcal{G}_V^{\hat{G}}$ with a high probability, or $\mathcal{Y}_{\hat{G}}(G_t)$ is close to $\lfloor |V|/|V_{\hat{G}}| \rfloor$ with high probability. In order to formalize these notions, we will utilize the concept of stochastic stability [50]. The application of stochastic stability to self-assembly is novel, although a similar notion has been applied to the analysis of gene regulatory networks [38]. A review of stochastic stability and the resistance tree method is provided in the mathematical preliminaries.

Before introducing our synthesis algorithms, we describe some desirable features of Φ and $\mathcal{R}$.

3.3.5 Recoverable states

We will be interested in Φ and $\mathcal{R}$ that maintain a natural relationship between the unlabeled graph (V_G, E_{G_t}) and the labeling function at time t, l_{G_t}. In particular, we would like to be able to generate the labeled graph from the unlabeled graph so that the two agree up to a label-preserving isomorphism. There are some clear benefits to this feature. First, insisting on such a degenerate internal state limits the freedom in specifying agent behavior in a manner appropriate for certain applications where agent homogeneity cannot be circumvented so easily. Second, Σ could be augmented (perhaps on a different time-scale) with periodic global information updates that can be used to correct any errors in the internal states.

Definition 3.3.5 *Given G_0 and F, we say that Φ and $\mathcal{R}$ produce a Σ with* ***recoverable states*** *if there exists $\tilde{l} : 2^{V_G \times V_G} \times V_{G_0} \to \mathcal{S}$ such that for any G that is observed with positive probability under Σ we have that (V_{G_0}, E_G, l_G) and $(V_{G_0}, E_G, \tilde{l}(E_G, \cdot))$ are a label-preserving isomorphism.*

We will be chiefly interested in the initial graph

$$G_0 = (\{1, 2, ..., N\}, \{\emptyset\}, l_0(\cdot) = s_0).$$

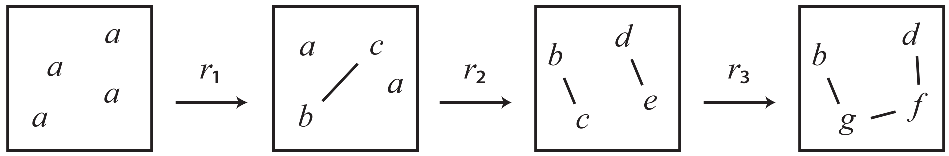

Figure 7: Successful realizations of $\{G_t\}$ entails construction of a four-node chain in parallel.

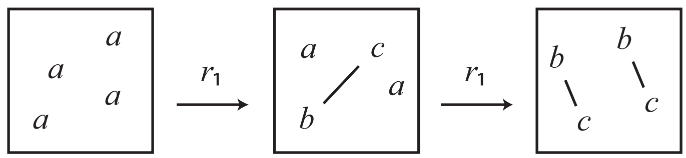

Figure 8: The system in Example 3.3.4 can also exhibit deadlock. The labels in the third graph pictured are not derivable from the set of edges alone.

Furthermore, since we consider only F bounded away from zero, definition 3.3.5 will be a property of Φ and $\mathcal{R}$ alone. While the definition may appear a bit cumbersome, it will be straightforward to see if a particular Φ and $\mathcal{R}$ produce recoverable states only.

Of the examples above, it is easy to verify that only Example 3.3.2 does not produce recoverable states. However, that example also violated our communication constraint as it utilized a mixed rule. Since we will be interested in algorithms synthesizing rules that obey both desiderata, we give another example.

Example 3.3.4 (Multiple applicable rules) *A straightforward way to generate Σ with non-recoverable states is to introduce rules with identical left hand sides. Consider the system $\Sigma = (G_0, F, \Phi, \mathcal{R})$ defined by*

$$G_0 = (\{1, 2, 3, 4\}, \{\emptyset\}, l_0(\cdot) = a)$$

$$F \sim \text{ i.i.d. uniform}$$

$$\Phi = \begin{cases} a \quad a \rightharpoonup b - c, & (r_1) \\ a \quad a \rightharpoonup d - e, & (r_2) \\ c \quad e \rightharpoonup f - g, & (r_3) \end{cases}$$

$$\mathcal{R}(\cdot) = 1.$$

Suppose $\hat{G} = (\{1, 2, 3\}, \{12, 23, 34\})$*; a chain of four vertices. Notice that* r_1 *and* r_2 *are both applicable when two singleton nodes with label a are selected. Figure 7 illustrates a successful trajectory, while Figure 8 illustrates the possibility of deadlock. Notice that the third graphs in each figure are identical apart from their labels, implying that the labels cannot be determined unambiguously from the unlabeled graphs.* §

One algorithm we will consider will synthesize rules that introduce non-recoverable states and it is precisely this phenomenon (multiple applicable rules) that will be responsible. A second feature we will strive for is uniqueness of labels in complete assemblies.

3.3.6 Uniqueness of final states

It is desirable for the agents participating in a complete assembly to know their role in that assembly. If they do not, then it is not possible for nodes sharing redundant labels to exhibit distinct behaviors. Self-assembly will often be only a part of a larger system objective. In this case if an agent's state does not imply their position in the final assembly unambiguously then system architects are limited in the diversity of behaviors they can induce in the self-assembled structures. Even if self-assembly is the lone objective, non-unique final states limit the composability of self-assembled structures in higher-level self-assembly processes. This restricts the level of parallelization and decentralization that is attainable. Motivated by these considerations we propose the following definition:

Definition 3.3.6 *Given* G_0, $\hat{G}$ *and* F*, we say that* Φ *and* $\mathcal{R}$ *produce a* Σ *with* ***unique final states*** *if there exists an injective labeling function* $\hat{l} : V_{\hat{G}} \to \mathcal{S}$ *such that any subgraph observed with positive probability that is isomorphic to* $\hat{G}$ *is a label-preserving isomorphism of* $\{V_{\hat{G}}, E_{\hat{G}}, \hat{l}\}$.

All of the examples we have encountered thus far produce unique final states. We give an example of a rule set that does not produce unique final states.

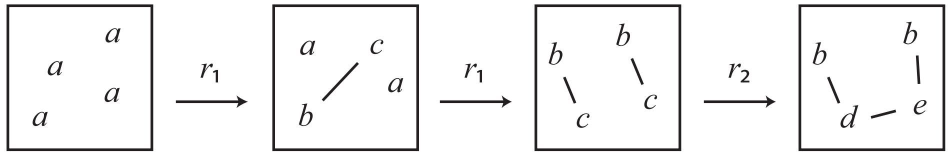

Figure 9: The labels in the fourth graph pictured are not unique.

Example 3.3.5 (Non-unique final states) *Consider the system* $\Sigma = (G_0, F, \Phi, \mathcal{R})$ *defined by*

$$G_0 = (\{1,2,3,4\}, \{\emptyset\}, l_0(\cdot) = a)$$

$$F \sim \textit{ i.i.d. uniform}$$

$$\Phi = \begin{cases} a \quad a \rightharpoonup b - c, & (r_1) \\ c \quad c \rightharpoonup d - e, & (r_2) \end{cases}$$

$$\mathcal{R}(\cdot) = 1.$$

Suppose $\hat{G} = (\{1,2,3\}, \{12, 23, 34\})$*; a chain of four vertices. Figure 9 illustrates a successful trajectory. Two agents have state* b *so the process does not give unique final states.* §

Next, we introduce our first algorithm, `Singleton`, which provides self-assembly performance guarantees and satisfies both constraints (constructive/destructive binary, reversibility) and at the same time guarantees both recoverable states and unique final states.

3.4 A serial algorithm

The `Singleton` algorithm generates a rule set Φ from a target graph $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$. In [30] we presented a nearly identical algorithm as a standalone system without the notation of graph grammars. We use numbers instead of letters for labels. The algorithm is a recursion.

In line 8, we use the reversible arrows, $\rightleftharpoons$, to indicate that the rule is to be understood as two rules; the second being the rule obtained by switching the left and right hand sides. Evident from line 8 is the reason behind the name: all constructive rules involve a node

Algorithm 1 `Singleton`(V, E, k, s)

$\Phi \leftarrow \{\emptyset\}$
if $|n_E(k)| = 0$ **then**
  return (s, Φ)
else
  $\{v_j : j = 1, 2, ..., |n_E(k)|\} \leftarrow n_E(k)$
  $\bar{s} \leftarrow s$
  for $j = 1$ to $|n_E(k)|$ **do**
    $\Phi \leftarrow \Phi \cup \{\bar{s} \quad 0 \rightleftharpoons (s+1) - (s+2)\}$
    $\bar{s} \leftarrow s + 1$
    $s \leftarrow s + 2$
    let (V^j, E^j) be the component of $(V, E - \{kv_j\})$ containing v_j
    $(s_j, \Phi_j) \leftarrow$ `Singleton`(V^j, E^j, v_j, s)
    $\Phi \leftarrow \Phi \cup \Phi_j$
    $s \leftarrow s_j$
  end for
end if
return (s, Φ)

with label 0—the label reserved for nodes not participating in any edges. To obtain a rule set, we run `Singleton`$(V_{\hat{G}}, E_{\hat{G}}, k, 0)$ for any $k \in V_{\hat{G}}$. The target graph $\hat{G}$ must be connected and acyclic. The algorithm is simple. We treat k as the root of the tree. The algorithm iterates through k's neighbors one at a time. Assuming $|V_{\hat{G}}| > 0$ the first rule is always

$$0 \quad 0 \rightleftharpoons 1 - 2,$$

where 1 is the label assigned to the node that will play the role of the root and 2 is the label of it's neighbor. If this neighbor has no other edges we proceed to the next neighbor and add the rule

$$1 \quad 0 \rightleftharpoons 3 - 4,$$

so that the node playing the role of k forms an edge with a singleton thereby filling a vacancy, and updates its label. Since the node playing the role of k has changed its label, there is no longer an applicable deconstructive rule between it and the node with label 2. We continue to proceed in this manner for each neighbor of the k-node. If one of the neighbors has neighbors other than k then we make a recursive call to `Singleton` treating

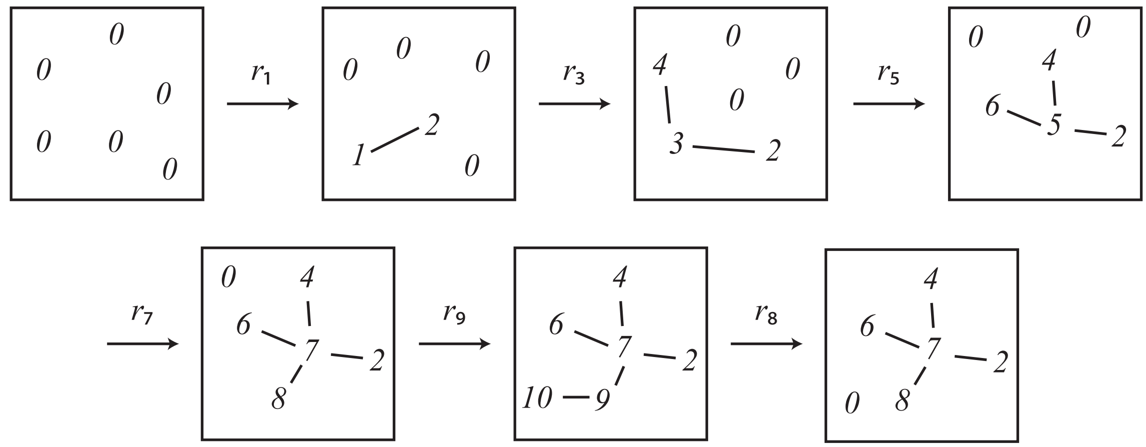

Figure 10: The system in Example 3.4.1 can assemble itself through application of the odd-numbered (constructive rules) in order, but may then apply destructive rules and disassemble.

the neighbor as the k-node (i.e. the root) of the graph obtained by making a cut between the original k-node and the neighbor. We keep track of the largest label s so that each new node added is assigned a unique label.

At a high-level, the algorithm succeeds because each singleton added on is able to determine its role in the target graph from the label of the node it forms an edge with. This information determines what vacancies, if any, it has for new nodes. The internal states thus provide only limited information about the overall structure of the subgraph that agents participate in. After a singleton joins up and receives a role, it may update its state as it fills vacancies, but will not know whether its neighbors have filled their vacancies. The rule set returned by `Singleton` is not necessarily immune to deadlock, but an appropriate choice of $\mathcal{R}$, the function assigning rule probabilities, will be accompanied by a strong performance guarantee. Next we provide a simple example of the `Singleton` algorithm.

Example 3.4.1 (`Singleton` algorithm) *Consider the target graph* $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ *defined by*

$$V_{\hat{G}} = \{1, 2, 3, 4, 5, 6\}$$

$$E_{\hat{G}} = \{12, 13, 14, 15, 56\}.$$

Let Φ be the rule set returned by `Singleton`*$(V_{\hat{G}}, E_{\hat{G}}, 1, 0)$*

$$\Phi = \begin{cases} 0 \quad 0 \rightleftharpoons 1-2, & (r_1, r_2) \\ 1 \quad 0 \rightleftharpoons 3-4, & (r_3, r_4) \\ 3 \quad 0 \rightleftharpoons 5-6, & (r_5, r_6) \\ 5 \quad 0 \rightleftharpoons 7-8, & (r_7, r_8) \\ 8 \quad 0 \rightleftharpoons 9-10. & (r_9, r_{10}) \end{cases}$$

Now consider a complete Σ as follows

$$G_0 = (\{1, 2, 3, 4, 5, 6\}, \{\emptyset\}, l_0(\cdot) = 0)$$

$$F \sim \textit{ i.i.d. uniform}$$

$$\mathcal{R}(\cdot) = 1.$$

Figure 10 illustrates an execution of this system that successfully assembles via the application of the constructive rules in order. Unfortunately, in the last graph the application of a destructive rule has taken the system out of $\mathcal{G}_V^{\hat{G}}$. §

3.4.1 Analysis of `Singleton`

A consequence of Φ being a reversible set of rules is that completed assemblies cannot be made stable— removing a part from a complete assembly and lowering the yield by one occurs with positive probability. This phenomenon was observed in the preceding example. However, the reversibility has the benefit of freeing the system from deadlock. Properly balancing these two attributes via $\mathcal{R}$ will be necessary in order to provide any sort of performance guarantee for `Singleton` systems.

Consider an arbitrary connected, acyclic $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ and the initial graph

$$G_0 = (\{1, 2, ..., N\}, \{\emptyset\}, l_0(\cdot) = 0)$$

with each $r \in \Phi$ having probability

$$\mathcal{R}(r) = \begin{cases} a_r, & r \text{ is constructive} \\ \epsilon, & r \text{ is destructive} \end{cases}$$

where Φ is obtained from $\texttt{Singleton}(V_{\hat{G}}, E_{\hat{G}}, k, 0)$ for any $k \in V_{\hat{G}}$. The values $a_r \in (0, 1]$ are arbitrary constants.

This Σ is a regular perturbed Markov process over the space of graphs, henceforth referred to as P^ϵ. In particular, each state of P^ϵ is an equivalence class of graphs that are isomorphic to each other. The unperturbed process, P^0, is obtained by removing the destructive rules from Φ and $\mathcal{R}$. The following result is immediate.

Lemma 3.4.1 *The absorbing states of P^0 are all states where each subgraph of G is isomorphic to a subgraph of $\hat{G}$. Either $|n_{E_G}(i)| \geq 1 \; \forall i \in V_G$ or there exists one $i \in V_G$ such that $|n_{E_G}(i)| = 0$.*

In other words, every assembly in every absorbing state of P^0 is a partial or complete assembly. The only circumstance were a node without any edges persists is when all other nodes participate in complete assemblies. Otherwise, the singleton node and some other node would comprise a left hand side of a constructive rule in Φ, which would contradict the state's being absorbing. Clearly P^0 has many states in $\mathcal{G}_V^{\hat{G}}$ as well as many states with nearly maximum yields, but also has quite a few deadlock states with low yields. The absorbing states are the only states we need to consider in determining the stochastically stable states of P^ϵ.

The performance guarantees for the `Singleton` algorithm rely on the following theorem:

Theorem 3.4.1 *The stochastically stable states of P^ϵ are the absorbing states of P^0 with the minimum number of disjoint maximal connected subgraphs. In particular, there are $\lfloor |V_{G_0}|/|V_{\hat{G}}| \rfloor$ such subgraphs in each and every one of the stochastically stable states.*

Proof: The proof is based on the construction of rooted trees for the claimed class of stochastically stable states and comparison with the trees corresponding to all other absorbing states. Let Z_0 be the absorbing states of P^0. We will partition Z_0 into disjoint sets Z_m where each state in Z_m has m assemblies, so that

$$\bigcup_{m \in \mathcal{M}} Z_m = Z_0, \mathcal{M} = \{\lceil |V_{G_0}|/|V_{\hat{G}}| \rceil, \lceil |V_{G_0}|/|V_{\hat{G}}| \rceil + 1, ..., \lfloor |V_{G_0}|/2 \rfloor\}.$$

The rooted trees for each absorbing state contain $|Z_0| - 1$ edges. There is nonzero resistance associated with each of these edges because the states are all absorbing. For P^ϵ, the resistance is at least one for each edge. We will show that a rooted tree satisfying this minimum resistance of $|Z_0| - 1$ can be constructed for each state in $Z_{\lceil |V_{G_0}|/|V_{\hat{G}}| \rceil}$.

To shorten the proof we restrict our interest to the case where both $N = |V_{G_0}|$ and $|V_{\hat{G}}|$ are even. A similar construction exists for the cases where either N, $|V_{\hat{G}}|$, or both are odd. Let $z_{N/2} \in Z_{N/2}$ be the state with all assemblies as pairs of nodes. Let $z_{N/2-1} \in Z_{N/2-1}$ be the state arrived at by applying the appropriate destructive rule on one of the pairs and then applying constructive actions to attach the two free atoms to one of the other pairs. We proceed like this for each $z_m \in Z_m$ letting z_{m-1} be arrived at by breaking up a pair and transferring the pieces to the largest possible assembly. This requires a destructive rule followed by two constructive rules. Figure 11 illustrates an example of this procedure.

We will first construct the tree rooted at $z_{\lceil N/|V_{\hat{G}}| \rceil}$. There are edges corresponding to the z_m as follows:

$$z_{\lfloor N/2 \rfloor} \rightarrow z_{\lfloor N/2 \rfloor - 1} \rightarrow ... \rightarrow z_{\lceil N/|V_{\hat{G}}| \rceil}.$$

Each edge $z_m \rightarrow z_{m-1}$ represents breaking up one assembly of two nodes (resistance of one), and then having those nodes form together. This will form the backbone of the tree. Figure 12 illustrates the concept we will use to complete the tree rooted at $z_{\lceil N/|V_{\hat{G}}| \rceil}$.

Each row contains the states with a particular number of assemblies. What remains to be shown is that all states in Z_m can reach z_m via a path through states in Z_m with all edges having resistance 1. Consider a state $y \in Z_m, y \neq z_m$. Each z_m consists of only two-node

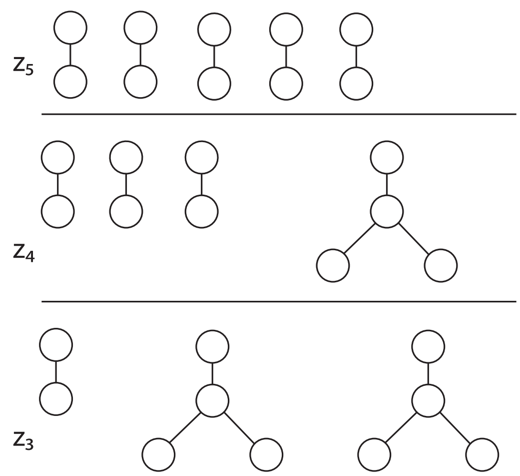

Figure 11: The states $z_m, m \in \{3,4,5\}$, $N = 10$, $|V_{\hat{G}}| = 4$.

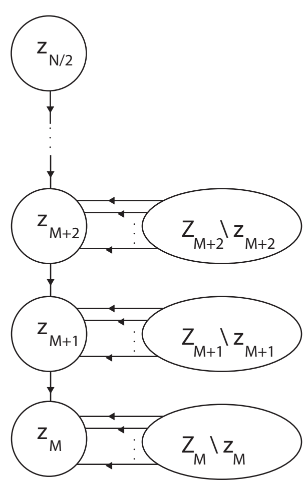

Figure 12: The structure of the tree rooted at $z_{\lceil N/|V_{\hat{G}}|\rceil}$; note that $M = \lceil N/|V_{\hat{G}}|\rceil$.

assemblies, completed assemblies, and at most one other assembly. Let x_m refer to the largest incomplete assembly of z_m (let it be a two-node assembly if that is the largest). We can construct the path to z_m for the two cases for y:

Case 1: There exists a maximal connected subgraph of y that is isomorphic to x_m. In this case, we take the smallest assembly with more than two nodes and shift a node to the largest incomplete assembly other than the one that is isomorphic to x_m. We continue this process until we obtain z_m. Each step in the process involves one destructive rule and therefore an edge with resistance one linking to a distinct absorbing state in Z_m.

Case 2: There is no maximal connected subgraph of y that is isomorphic to x_m. In this case, we take the smallest assembly with more than two nodes and shift a node to the largest maximal connected subgraph that is isomorphic to a subgraph of x_m. We proceed like this until we obtain a maximal connected subgraph that is isomorphic to x_m, and then continue as in case 1.

We can repeat this process until we have covered all of the states in Z_m, avoiding any redundancies so that we form precisely $|Z_m|-1$ edges. Once we have applied this technique for all m we have obtained a rooted tree with each edge having resistance one so that $z_{\lceil N/|V_{\hat{G}}|\rceil}$ is stochastically stable.

This construction can be extended for all the other states in $Z_{\lceil N/|V_{\hat{G}}|\rceil}$. For an arbitrary state $z' \in Z_{\lceil N/|V_{\hat{G}}|\rceil}, z' \neq z_{\lceil N/|V_{\hat{G}}|\rceil}$ we construct the tree just as above for the states in the sets $Z_m, m > \lceil N/|V_{\hat{G}}|\rceil$ and for $z_{\lceil N/|V_{\hat{G}}|\rceil}$. Then we insert the edges between $z_{\lceil N/|V_{\hat{G}}|\rceil}$ and z' in the same way as above except that the directions are reversed. Then we apply the exact same procedure as above for the remaining states in $Z_{\lceil N/|V_{\hat{G}}|\rceil}$, again avoiding any redundancies. These trees will also all have resistance one at every edge so that all of $Z_{\lceil N/|V_{\hat{G}}|\rceil}$ is stochastically stable.

For any state in $Z_m, m \neq \lceil N/|V_{\hat{G}}|\rceil$ the rooted trees all must include an edge that goes from a state with a smaller number of assemblies to a state with a larger number of assemblies. This can only be accomplished by application of two consecutive destructive rules corresponding to an edge with resistance two. Since all other edges are at best resistance one, all of these rooted trees have resistance equal to at least $|Z_0|$. We therefore conclude that the stochastically stable states are precisely $Z_{\lceil N/|V_{\hat{G}}|\rceil}$, the states with the minimum number of assemblies. ■

Unfortunately, not all stochastically states of P^ϵ are in $\mathcal{G}_V^{\hat{G}}$. However, there is a significant exception.

Corollary 3.4.1 *If $N = m|V_{\hat{G}}|$ for some $m \in \mathbb{Z}$ then all stochastically stable state have all maximal connected subgraphs of G isomorphic to $\hat{G}$.*

This result is immediate from the preceding Theorem. Of course, such a strong result will not hold when N is not an integer multiple of $|V_{\hat{G}}|$. This leads to the following curiosity: We can decrease the minimum yield among the stochastically stable states by increasing N. This is somewhat surprising given that `Singleton` generates rules without any consideration of N, and increasing N makes more parts available for assembly. This phenomenon leads to weak performance for `Singleton` when N is both not much larger than $|V_{\hat{G}}|$ and not an integer multiple of $|V_{\hat{G}}|$. Nevertheless, the situation is much better when N is large.

Theorem 3.4.2 *All stochastically stables states of P^ϵ have no more than $(|V_{\hat{G}}| - 1)^2$ nodes not part of a connected subgraph isomorphic to $\hat{G}$. Further, at most $|V_{\hat{G}}| - 1$ subassemblies are incomplete.*

Proof: Let $N = (|V_{\hat{G}}| - 1)^2$. The maximum number of incomplete assemblies is $|V_{\hat{G}}| - 1$ assemblies with $|V_{\hat{G}}| - 1$ nodes in each assembly. Each increase of N by one, must add one complete assembly and reduce the number of nodes not participating in complete assemblies by $|V_{\hat{G}}| - 1$. This continues until we reach $N = |V_{\hat{G}}|(|V_{\hat{G}}| - 1)$ and there are zero nodes not part of complete assemblies in the stochastically stable states. This process repeats for $|V_{\hat{G}}|(|V_{\hat{G}}| - 1) + 1$ through $|V_{\hat{G}}|^2$ so that it is easy to show by induction that $(|V_{\hat{G}}| - 1)^2$ is always the maximum number of nodes not part of complete assemblies. ■

3.4.2 Remarks

Theorem 3.4.2 upper bounds the number of reject assemblies for all N. When $N \gg |V_{\hat{G}}|$ the yield of `Singleton` is only negligibly different from the maximum. It is an open question as to whether or this guarantee can be improved upon without compromising on the constraints or features of the internal states. Empirically, we have found that introducing resistances greater than one for destructive rules applied further from the k-node can improve performance in some simulations, but these results are yet to be formalized.

Another feature of `Singleton` is that parts are only added one-at-a-time. In Appendix 3.9 we present a non-reversible version of `Singleton` with non-recoverable states that converges to $\mathcal{G}_V^{\hat{G}}$ almost surely. Below we will present a process that provides the same performance guarantee, so this modified `Singleton` process is of interest only if one-at-a-time assembly is preferable.

The basic action of the `Singleton` process is to place more probability weight on assembly than disassembly. The system tends toward assembly because of this. At the same time, the positive probabilities associated with disassembly alleviate deadlock. The shortcomings of the `Singleton` process are related to the fact that it views each edge the same way. This is why when we have

$$|V_{\hat{G}}| < N < 2|V_{\hat{G}}|$$

the probability of observing two incomplete assemblies is comparable to the probability of observing one complete assembly and one incomplete assembly. Both situations exhibit the same number of edges and it is the number of edges that the process drives down, or equivalently, the total number of assemblies.

Next we consider an algorithm for synthesizing Φ when non-recoverable states are allowed. We will see that this process is able to improve upon the performance of the `Singleton` process by treating some edges differently than others.

3.5 A parallel algorithm

Under random pairwise comparison dynamics the sequence of graphs $\{G_t\}$ is random. The strongest possible performance guarantee that can be provided is that $\{G_t\}$ converges to the set $\mathcal{G}_V^{\hat{G}}$ almost surely. Equivalently, we can be arbitrarily close to certain that the system will be assembled if we wait long enough to make our observation. If we impose that Φ is reversible, then this sort of guarantee is not possible. In the previous section we introduced an alternative for this circumstance in the form of stochastic stability. Stochastic stability provides a continuum of systems such that the stationary probability of observing a mostly complete $\{G_t\}$ goes to one as the parameter ϵ goes to zero. We would like to be able to improve upon the result of the previous section. In particular, we would like the stochastically stable states to be a subset of $\mathcal{G}_V^{\hat{G}}$. We will present an algorithm that accomplishes this feat while observing the constraints that Φ be a reversible set of binary constructive and deconstructive rules, but will in most cases introduce non-recoverable states. The process will have unique final states, however. If, in addition, an irreversible rule is allowed, then the system will converge to $\mathcal{G}_V^{\hat{G}}$ almost surely.

As we saw in Example 3.3.4, non-recoverable states are often associated with non-uniqueness of the left hand sides in Φ. The `Lynchpin` algorithm will generate non-recoverable states precisely because of this issue. Like, `Singleton`, `Lynchpin` is a recursion that generates Φ from a target graph $\hat{G}$ and an initial node k.

To obtain a rule set we run `Lynchpin`$(V_{\hat{G}}, E_{\hat{G}}, k, 0)$ for any $k \in V_{\hat{G}}$. The target graph $\hat{G}$ must be connected and acyclic. The defining feature of rule sets generated from `Lynchpin` is the presence of a completing rule. That is, every assembly is completed by application of the same rule. In order to highlight this feature, we next consider an example $\hat{G}$ and compare the rule sets generated by `Lynchpin` and `Singleton`.

Example 3.5.1 (Completing rules) *Suppose* $\hat{G} = (\{1, 2, 3\}, \{12, 23, 34\})$*; a chain of four*

Algorithm 2 $\texttt{Lynchpin}(V, E, k, s)$

$\{v_j : j = 1, 2, ..., |n_E(k)|\} \leftarrow n_E(k)$
for $j = 1$ to $|n_E(k)|$ **do**
  if $|n_E(v_j)| \geq 2$ **then**
    let (V^j, E^j) be the component of $(V, E - \{kv_j\})$ containing v_j
    $(s_j, \Phi_j) \leftarrow \texttt{Lynchpin}(V^j, E^j, v_j, s)$
    $s \leftarrow s_j$
  else
    $s_j \leftarrow 0$
    $\Phi_j \leftarrow \{\emptyset\}$
  end if
end for
$\Phi \leftarrow \Phi_1 \cup \{s_1 \quad 0 \rightleftharpoons (s+1) - (s+2)\}$
$s \leftarrow s + 2$
for $j = 2$ to $|n_E(k)|$ **do**
  $\Phi \leftarrow \Phi \cup \Phi_j \cup \{s_j \quad s \rightleftharpoons (s+1) - (s+2)\}$
  $s \leftarrow s + 2$
end for
return (s, Φ)

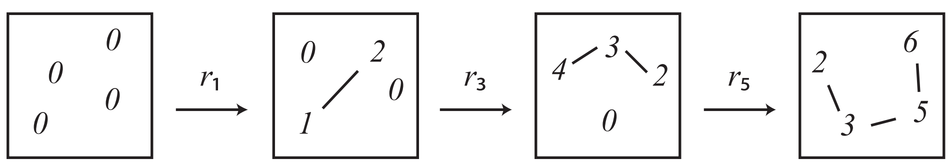

Figure 13: Assembly of $\hat{G}$ via Φ_S with r_5 being the last rule applied, clearly the order of r_3 and r_5 can be reversed.

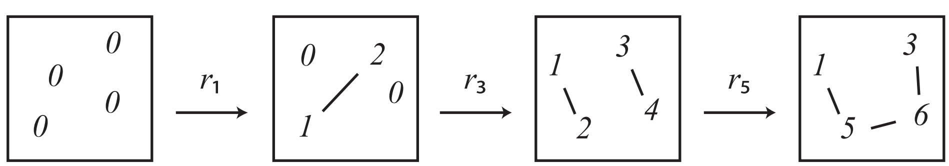

Figure 14: Assembly of $\hat{G}$ via Φ_L always culminates with r_5.

vertices, as in Example 3.3.4. Furthermore let

$$\Phi_S = \texttt{Singleton}(V_{\hat{G}}, E_{\hat{G}}, 2, 0) = \begin{cases} 0 & 0 \rightleftharpoons 1-2, \quad (r_1, r_2) \\ 1 & 0 \rightleftharpoons 3-4, \quad (r_3, r_4) \\ 4 & 0 \rightleftharpoons 5-6, \quad (r_5, r_6) \end{cases}$$

and

$$\Phi_L = \texttt{Lynchpin}(V_{\hat{G}}, E_{\hat{G}}, 2, 0) = \begin{cases} 0 & 0 \rightleftharpoons 1-2, \quad (r_1, r_2) \\ 0 & 0 \rightleftharpoons 3-4, \quad (r_3, r_4) \\ 2 & 4 \rightleftharpoons 5-6, \quad (r_5, r_6) \end{cases}$$

Figure 13 shows one trajectory for Φ_S. In this case, the process culminates in application of r_5. However, we can also reverse the order of r_5 and r_3. The consequence is that there is no unique completing rule. For this example we could have chosen the starting node argument k differently in the `Singleton` algorithm and generated a rule set with a unique completing rule, but it is easy to construct $\hat{G}$ for which no such choice exists.

Rule sets generated by the `Lynchpin` algorithm always give self-assembly trajectories that culminate in a unique completing rule. This is true irrespective of the starting node k. Figure 14 illustrates this phenomenon for our present Example. It is this feature of the `Lynchpin` algorithm that will enable us to improve upon the guarantees for the `Singleton` algorithm. §

The `Singleton` algorithm does not have a unique completing rule because self-assembly proceeds outward from the starting node k. Since the target graph likely has many branches, any of a number of leaves can be added on last. In contrast, `Lynchpin` assembles from each leaf in towards the k-node so that the overall process culminates with two sub-graphs joining together. These subgraphs are themselves assembled recursively in the same manner.

Recall that the principal action of `Singleton` is to seek absorbing states with a minimum number of edges. This process allows up to $|V_{\hat{G}}| - 1$ incomplete assemblies. When N

is not large this can be a significant limitation. We will see that `Lynchpin` can easily circumvent this limitation due to the presence of a completing rule. In particular, suppression of the complement of the completing rule is all that is needed.

Let $\hat{s}$ be the label returned by `Lynchpin`. Then there is one rule whose left hand side contains this label— the complement of the completing rule, $\hat{r}$. We will be interested in the following rule probabilities

$$\mathcal{R}(r) = \begin{cases} a_r, & r \neq \hat{r} \\ \epsilon, & r = \hat{r} \end{cases}$$

where $a_r \in (0, 1]$ are arbitrary constants. As with the `Singleton` algorithm, this choice of $\mathcal{R}$ gives a regular perturbed Markov process, P^{ϵ}. The unperturbed process P^0 is obtained by removing $\hat{r}$ from Φ.

3.5.1 Analysis of `Lynchpin`

Before we analyze the random process induced by Φ for our choice of $\mathcal{R}$, we will establish some properties of Φ. First, we show that the rule set returned by `Lynchpin` is, in principle, capable of constructing $\hat{G}$.

Lemma 3.5.1 *For any tree $\hat{G}$, let Φ be given by `Lynchpin` $(V_{\hat{G}}, E_{\hat{G}}, k, 0)$. Then there exists a sequence of constructive actions in Φ that, applied to $G_0 = \{V_{\hat{G}}, \{\emptyset\}, l_{G_0}(\cdot) = 0\}$ in succession, result in $\hat{G}$.*

Proof: The proof is by induction on the depth of the tree rooted at k. If k has no neighbors then the algorithm returns no rules and the lemma is satisfied vacuously. The base case is a depth of 1. In this case, $\hat{G}$ is a star with k at its center. Line 1 assigns any order to the neighbors of k. Lines 2-11 iterate through these neighbors and in this case always execute lines 8 and 9 that assign $s_j = 0$ and $\Phi_j = \{\emptyset\}$ for each neighbor v_j because, by assumption, each v_j has no neighbors other than k. Line 12 gives the first rules $\{0 \quad 0 \rightleftharpoons 1 - 2\}$. The part assigned state 2 will continue on with the role of k. Each rule added in lines 14-17 adds another singleton to k. The lemma is satisfied by applying the constructive rules from Φ in the order that they were added.

For the induction step assume that `Lynchpin` satisfies the lemma (by applying the constructive rules in the order they were added to Φ) when the depth of the tree rooted at k is at most D. Now suppose that the depth of the tree rooted at k is $D + 1$. Now we will expect to see some of k's neighbors having neighbors other than k. In this case we make the recursive call to `Lynchpin` on line 5 with v_j as the new k-node. Since the depth of the subtree rooted at v_j obtained in line 4 is at most D, we get a sequence of rules that build this subtree by assumption. We note that s_j is the state of v_j in the completed subtree. We also note that each recursive call introduces only unused labels. Line 12 gives a rule that adds k as a singleton to the completed subtree for v_1. Lines 14-17 now add the remaining subtrees to k. If a subtree is just v_j then $s_j = 0$ and we add a singleton to k. If the subtree for v_j is not a singleton then s_j is the state of v_j once that subtree has finished assembling. The lemma is once again satisfied by applying the constructive rules from Φ in the order that they were added, completing the proof. ■

The convergence proof depends on one additional property of `Lynchpin`— the presence of a unique *completing rule*.

Lemma 3.5.2 *For any tree $\hat{G}$, let Φ be given by* `Lynchpin` $(V_{\hat{G}}, E_{\hat{G}}, k, 0)$*. Let G be the labeled graph obtained by applying the constructive rules in Φ (in the order they were added) to $G_0 = \{V_{\hat{G}}, \{\emptyset\}, l_{G_0}(\cdot) = 0\}$. Then there is only one rule in Φ applicable to G and it is deconstructive and involves k. Furthermore none of the labels in G appear in the left hand sides of the constructive rules in Φ.*

Proof: The proof is again by induction on the depth of the tree rooted at k. For the base case we examine lines 12-17. Each rule adds a singleton to k and increases k's state by 2. Since k's state has changed only the last deconstructive rule added applies. The final state of k is also greater than any of the states on the left hand sides of the constructive rules.

For the inductive step assume the lemma holds for trees rooted at k with depth no greater than D and suppose that the tree rooted at k has depth $D + 1$. Then lines 2-11 give rules that produce subtrees satisfying the lemma. Now, just as in the base case, when we add these subtrees to k we increase v_j's state so that it no longer has any applicable rules unless it was the most recent addition. The final state of k is also again greater than any of the left hands sides of constructive rules. Only one rule applies to the finished product and it is the deconstructive rule severing k and $v_{|n_E(k)|}$— the lynchpin.■

The requirement on the left hand sides of constructive rules is to ensure that the assembly will not "overassemble" in the presence of additional parts. We would like the labels of complete assemblies arrived at through Φ to be unique so that the above lemma applies to all complete assemblies. It turns out that this is indeed the case.

Lemma 3.5.3 *For any tree $\hat{G}$, let Φ be given by* `Lynchpin` $(V_{\hat{G}}, E_{\hat{G}}, k, 0)$. *Let G be any complete assembly obtained by applying constructive rules in Φ to $G_0 = \{V_{\hat{G}}, \{\emptyset\}, l_{G_0}(\cdot) = 0\}$. Then G is a label-preserving isomorphism of the graph obtained in Lemma 3.5.2.*

Proof: The proof is once again by induction on the depth of the tree rooted at k. The base case of a star with center k gives constructive rules that must be applied in the exact order they were added. This implies that the final labels are unique. For the inductive step assume the labels are unique for depth D. When the depth is $D + 1$ the recursive calls give, by assumption, subtrees with unique labels. Then, similar to the base case, the subtrees must be combined in a specific order so that the final labels are again unique. ■

Note that the different subtrees can be completed in any order so that `Lynchpin` gives rules which allow for parallel self-assembly. The rules do not need to be applied in exactly the order they were added to Φ (as in Lemma 3.5.2). With these three lemmas in hand we proceed toward our main result, the performance guarantees for random pairwise selection using rule sets generated by `Lynchpin`.

While the unperturbed process in the case of `Singleton` was especially deadlock-prone, this is not true of the unperturbed process in this case. In fact, the stationary distribution of P^0 places positive probability on states in $\mathcal{G}_V^{\hat{G}}$ only. Of course, P^0 is not reversible. Nevertheless, we will first establish the performance guarantee for P^0, since the analysis for P^ϵ will be a straightforward extension of that result.

Consider an arbitrary connected, acyclic $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ and the initial graph

$$G_0 = (\{1, 2, ..., N\}, \{\emptyset\}, l_0(\cdot) = 0).$$

Φ and $\mathcal{R}$ are as specified in the previous section which gives the unperturbed process $\{G_t\}$. Recall that $\mathcal{Y}_{\hat{G}}(G_t)$ is the yield of $\hat{G}$ for the process at time t.

Lemma 3.5.4 *For the unperturbed `Lynchpin` process, $\mathcal{Y}_{\hat{G}}(G_t)$ is nondecreasing in t.*

Proof: Suppose for the sake of contradiction that there exists $\tau > 0$ such that $\mathcal{Y}_{\hat{G}}(G_\tau) < \mathcal{Y}_{\hat{G}}(G_{\tau-1})$. The only way that the number of maximal connected subgraphs of $G_{\tau-1}$ that are isomorphic to $\hat{G}$ can decrease is if $\hat{r}$ is applied, but this contradicts $\hat{r} \notin \Phi$ for the unperturbed process. ■

Next we establish that $\mathcal{Y}_{\hat{G}}(G_t)$ increases with positive probability.

Lemma 3.5.5 *Suppose that $\mathcal{Y}_{\hat{G}}(G_t) < \lfloor N/|V_{\hat{G}}| \rfloor$, then there exists a length of time T and a probability $p > 0$ such that $\mathbf{Pr}\left[\mathcal{Y}_{\hat{G}}(G_{t+T}) > \mathcal{Y}_{\hat{G}}(G_t)\right] = p$.*

Proof: Since we only need $p > 0$ we need only find one trajectory with positive probability. If there are $|V_{\hat{G}}|$ nodes with label 0 then we can select appropriate nodes and apply constructive rules. If there are insufficient nodes then destructive rules can be applied to incomplete assemblies to free up parts. In either case, the associated probability is positive and T is simply the number of rules applied. ■

The following result is now immediate.

Theorem 3.5.1 *For the unperturbed `Lynchpin` process, $G_t \to \mathcal{G}_V^{\hat{G}}$ almost surely.*

A subset of $\mathcal{G}_V^{\hat{G}}$ is the only recurrent class of the process. It follows that these are precisely the stochastically stable states of the perturbed process.

Theorem 3.5.2 *The stochastically stable states of the perturbed* `Lynchpin` *process are a subset of* $\mathcal{G}_V^{\hat{G}}$.

It is interesting to note that the unperturbed `Lynchpin` process utilizes just a single irreversible rule, yet provides the strongest possible form of performance guarantee. When the complement of this irreversible rule is introduced as a perturbation, we get the best possible form of performance guarantee for a reversible self-assembly process. Note that it may be possible to devise processes with better convergence rates, but the form of the performance guarantee would be the same.

3.6 Conservatism of Completing Rules

The `Lynchpin` gives unique final labels. However, unlike `Singleton` the states are not recoverable. That is, we cannot always infer the correct labels (up to a label-preserving isomorphism) from the unlabeled subgraph. The implication is that the agents' states are not auxiliary. Each agents behavior depends on more than just the structure of the assembly that it is participating in. The `Lynchpin` algorithm will, in general, produce several rules of the form $\{0 \quad 0 \rightleftharpoons x - y\}$ with different x, y for each rule. Consequently, the labels of the resulting subgraphs cannot be inferred from the associated unlabeled subgraphs.

While `Singleton` has both of the aforementioned features, it only gives an asymptotically maximum yield (in $|V_{G_0}|$). We achieve a maximum yield in `Lynchpin`, but sacrifice the feature of internal states being derivable from the unlabeled graph. It is an open question whether any reversible algorithm obeying the communication constraints can satisfy both desiderata and give maximum yields. However, we will show that if such an algorithm exists, it cannot exploit the notion of a completing rule. The proof is by way of a counterexample. First, we prove a lemma that gives a condition on assembly trajectories that any rule set exhibiting a completing rule must possess.

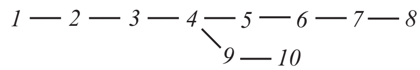

Figure 15: The counterexample is a tree that gives branches with different lengths greater than 1 from any root.

Lemma 3.6.1 *Suppose there is a sequence of constructive rules from a reversible, binary constructive/deconstructive rule set that produce $\hat{G}$ when applied successively to $G_0 = \{V_{\hat{G}}, \{\emptyset\}, l_{G_0}(\cdot) = 0\}$. Also assume that the complete assembly has only one applicable deconstructive rule and none of its labels appear in the left hand sides of the constructive rules. Then every non-singleton node that forms an edge must have also participated in the last edge that its subassembly formed.*

Proof: The proof is by contradiction. Suppose that there exists a constructive rule in the sequence that adds an edge to a node that was not among the last in its subassembly to form an edge. Reversibility demands that at least one deconstructive rule is applicable to the subassembly initially, the complement of the last constructive rule applied. If we then apply a constructive rule that does not involve either of the nodes whose states were set when the last rule was applied to that subassembly then there will be two applicable deconstructive rules. Each successive constructive rule application renders at most one deconstructive rule inapplicable. It follows that the final assembly has two or more applicable deconstructive rules, a contradiction ■

With this lemma we can prove our theorem on the conservatism of completing rules.

Theorem 3.6.1 *Any algorithm that gives reversible, binary constructive/deconstructive rule sets with completing rules must, for some target trees, either introduce states that cannot be determined from the unlabeled graph or give complete assemblies with non-unique states.*

Proof: The proof is by counterexample. Consider the graph in Fig. 15 We will give the argument for one particular choice of the completing rule. The arguments for all the other choices are similar. Suppose that the complement of the completing rule severs the edge

Table 1: Parameters for simulations of self-assembly algorithms.

parameter	value	comment
N	14	total number of parts
$V_{\hat{G}}$	{1,2,3,4}	target assembly has four parts
$E_{\hat{G}}$	{12,13,14}	see Figure 11
G_0	$(\{1, 2, ..., 14\}, \{\emptyset\}, l_0(\cdot) = 0)$	standard initial conditions
ϵ	.01	for `Lynchpin` and `Singleton`
a_r	1	see below
T	10,000	total number of samples for each simulation
F	uniform	agents are selected uniformly
n	100	number of simulations per algorithm

7 – 8. Then, by the preceding lemma, the next applicable deconstructive rule most sever 6 – 7, followed by 5 – 6 and 4 – 5. This gives a chain of six nodes. The next deconstructive rule severs either 3 – 4 or 4 – 9. In the first case we get chains of length 3, in the second case we get chains of lengths two and four. In any event, proceeding like this we see that 1 – 2 and 9 – 10 are two subassemblies that must appear in any trajectory that completes in the manner we have assumed. These two subassemblies are isomorphic to each other so in order to satisfy our desiderata they must have the same states. Also, they must have the same state participate in future edges because otherwise each will have a state appearing in the left hand side of a constructive rule at the end. It follows that the states corresponding to nodes 1 and 10 will be identical in the final assembly. ■

We could have used a smaller counterexample, but we would still be able to get unique states up to an isomorphism. Our counterexample shows even that is not always possible.

3.7 Simulations

The performance guarantees described for the algorithms in this paper are analytical. We present the results of simulations not to assert convergence (as this has already been done), but to compare performance between algorithms and to comment on transient behavior. The table summarizes the parameters used in the simulations.

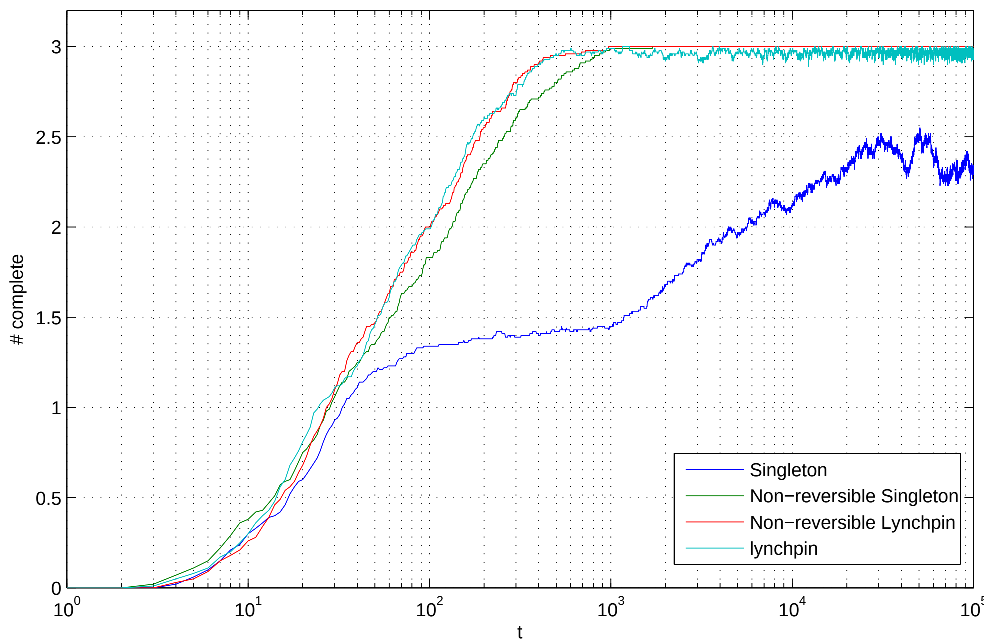

Figure 16: The maximum yield of three is eventually reached and maintained in all simulations of the two non-reversible processes. For `Lynchpin` the system lingers around three, while for `Singleton` it lingers between two and three.

We ran 100 simulations for each of the four algorithms: `Lynchpin`, `Singleton`, non-reversible `Lynchpin` (i.e. $\epsilon = 0$), and non-reversible `Singleton` (described in Appendix 3.9). We used $\mathcal{R}(r) = a_r = 1$ in each algorithm except for the rules in `Lynchpin` and `Singleton` that depend on ϵ. The results of the simulations are displayed in Figure 16, where we have averaged over the 100 simulations for each algorithm.

The maximum yield for this simulation is three assemblies. The set of states with three complete assemblies is invariant under the two non-reversible processes. This is consistent with the results of the simulation—all runs for both methods increase monotonically to the maximum yield within 2000 iterations and remain there. Since the two reversible processes do not exhibit such an invariance, both frequently contain fewer than three complete assemblies throughout the simulation. In the case of `Singleton`, observation of three complete assemblies is even less likely than observing fewer. Recall that, from Theorem 3.4.2, we expect no more than $(|V_{\hat{G}}| - 1)^2 = 9$ agents to fail to participate in complete assemblies in stochastically stable states of `Singleton`. Based on this bound we would expect to see only one complete assembly for this application of `Singleton`. The fact that we are almost always observing two or three assemblies highlights two deficiencies in the tightness of Theorem 3.4.2 as a bound on performance. First, the tightness of Theorem 3.4.2 is a function of N and $|V_{\hat{G}}|$. When N is an integer multiple of $|V_{\hat{G}}|$, stochastically stable states all have maximum yield and Corollary 3.4.1 is the relevant bound. Theorem 3.4.2 is a worst-case bound based on Theorem 3.4.1. When $N \gg |V_{\hat{G}}|$ the difference between the worst-case and N-dependent bounds are small. Since $N = 14$ is not very large, the bound's lack of tightness is readily apparent. It is straightforward to verify that Theorem 3.4.1 implies that, in this example, all of `Singleton`'s stochastically stable states have at least two complete assemblies. Second, the minimum number of complete assemblies among stochastically stable states is not the only number we should expect to regularly observe. In our simulations, observation of three complete assemblies was nearly as likely as observation of two. The theory of stochastic stability says only that as ϵ goes to zero

Table 2: Proportion of running time with each possible number of complete assemblies.

algorithm	0	1	2	3	mean
`Singleton`	.0080	.0318	.5964	.3638	2.3161
Non-reversible `Singleton`	.0002	.0006	.0023	.9969	2.9958
`Lynchpin`	.0002	.0006	.0346	.9646	2.9635
Non-reversible `Lynchpin`	.0002	.0005	.0014	.9979	2.9970

we should expect the states that are not stochastically stable to be observed with vanishing frequency. Some stochastically stable states can be much more frequently observed than others. This is why characterization of the set of stochastically stable states alone fails to give a tight bound on performance.

The proportion of time with each number of complete assemblies and the long-run average number of complete assemblies are summarized for each algorithm in the above table.

3.8 Discussion

We introduced a stochastic system framework for comparing the performance of different rule sets. We restricted ourselves to binary constructive and deconstructive rules as a communication constraint. We also insisted on reversibility, recoverable states, and unique final states. For this framework we presented the `Singleton` algorithm that could synthesize rules for any connected acyclic target and provide a performance guarantee in the form of a bound on the number of reject assemblies among stochastically stable states. We then relaxed the constraint on recoverable states and presented the `Lynchpin` algorithm that could synthesize rules for any connected acyclic target and provide a guaranteed maximum yield in the form of stochastic stability. We also showed that the maximum yield could be made an invariant of the system if even one irreversible rule is allowed.

The matter of whether or not a stronger performance guarantee can be made when recoverable states and unique final states are required remains an open question. We have seen some success in simulations by choosing different resistances for the deconstructive rules in the `Singleton` process so as to reduce the relative probability of disassembling more developed assemblies. Nevertheless, a rigorous analysis of such processes and an algorithm for finding these resistances for general $\hat{G}$ remain elusive.

We consider the primary contribution of this paper to be the existence results for self-assembly procedures under each set of constraints. The specific forms of `Singleton` and `Lynchpin` may or may not be especially relevant to self-assembly outside of their usefulness for establishing possibility results. We also suggest that stochastic stability is relevant to self-assembly, particularly when there is an interest in reversibility.

3.9 The non-reversible `Singleton` algorithm

Here we suggest a modification to the rules generated by the `Singleton` algorithm. With these rules, the system will converge to $\mathcal{G}_V^{\hat{G}}$ almost surely. However, the process will no longer be reversible or exhibit recoverable states. Only destructive rules are changed so that the system still assembles one part a time. We include this version of the algorithm for the case in which one-at-a-time assembly is preferable.

As before we begin by considering a connected acyclic target graph $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ and let Φ_S be the constructive rules from evaluation of `Singleton`. Let $\mathcal{L}$ be the set of all the labels in Φ_S. We also define $\mathcal{M} \subset \mathcal{L}$ as the set of final labels. That is, the labels that parts participating in completed assemblies can be assigned. A part with a label in $\mathcal{M}$ may or may not be participating in a complete assembly, but parts with labels in $\mathcal{L} \backslash \mathcal{M}$ do not participate in complete assemblies. Since each creation of an edge in `Singleton` assigns new states to both vertices, the number of edges a node participates in, or its degree, can be inferred from its label. We define $d : \mathcal{L} \to \mathbb{R}_+$ to be precisely this mapping. We also define $d_{\max} = \max_{s \in \mathcal{L}} d(s)$, the maximum degree.

Now we can define Φ with the appropriate deconstructive rules.

$$
\begin{aligned}
\Phi &= \Phi_S \cup \{\{s - s' \rightharpoonup D_{\hat{d}(s)-1} \quad D_{\hat{d}(s')-1}\} : s \in \mathcal{L}\backslash\mathcal{M},\ s' \in \mathcal{L}\} \\
&\cup\{\{D_n - s \rightharpoonup D_{n-1} \quad D_{\hat{d}(s)-1}\} : n \in \{1, 2, ...d_{\max}\},\ s' \in \mathcal{L}\} \\
&\cup\{\{D_n - D_m \rightharpoonup D_{n-1} \quad D_{m-1}\} : n, m \in \{1, 2, ...d_{\max}\}\}.
\end{aligned}
$$

We add two types of rules. First, we add rules that eliminate any edge involving a non-final label, with each node adopting a D (deficient) label. The subscript on the D label indicates the number of remaining edges for that node. Second, we add rules that eliminate any edge involving a D label, both nodes subsequently adopt the appropriate D labels. We define $D_0 \equiv 0$, so that a D node returns to normal once all its edges are severed. There will be rules in Φ that are never applicable, but we do not refine the rule set here.

We do not need to constrain $\mathcal{R}$ at all. The non-reversible `Singleton` process, like the (also non-reversible) unperturbed `lynchpin` process does not ever reduce the number of complete assemblies. This is because all the nodes in a complete assembly have final labels. Furthermore, any incomplete assembly must have at least one non-final label, so deadlock is avoided.

The non-reversible `Singleton` is similar to the non-reversible `lynchpin` process. The non-reversible `lynchpin` process never breaks up a complete assembly because it has no complement for the unique completing rule. The non-reversible `Singleton` process has no complements for any of the non-unique completing rules. This alone would lead to severe deadlock issues, so the additional deconstructive rules are added so that incomplete assemblies can disassemble. These rules allow the assembly to disassemble beginning at any node that is not final. Such nodes are essentially aware of the missing edges. The original `Singleton` rules only disassembled beginning from extremities (nodes with a single edge). Unfortunately, it is possible for an incomplete assembly to have the complement of a completing rule as the only applicable deconstructive rule. The non-reversible `Singleton` process allows disassembly beginning at nodes that are not extremities. It follows that

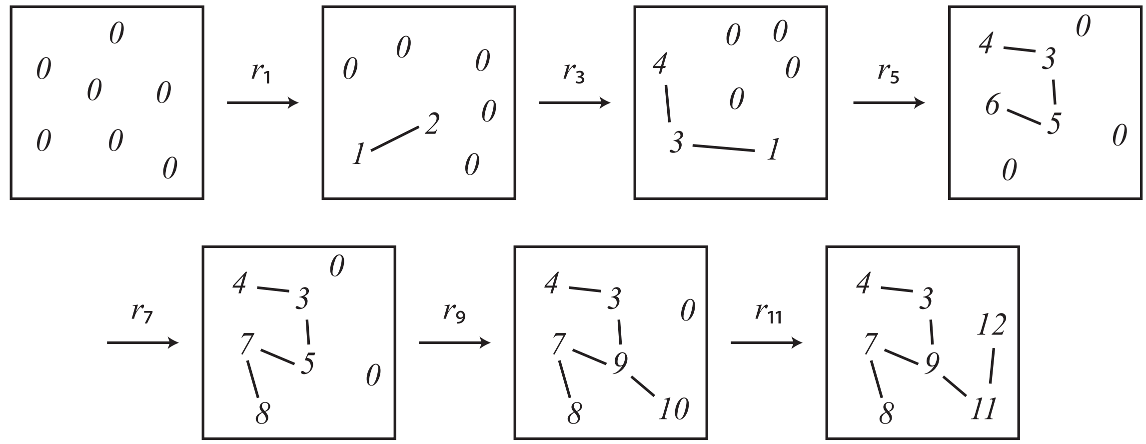

Figure 17: The presence of multiple appendages gives multiple completing rules for any starting node k.

unlike assembly, disassembly need not occur one-at-a-time.

Example 3.9.1 (Multiple appendages) *The distinction between the* `Singleton` *process and its non-reversible modification are evident when considering assemblies with multiple appendages. Consider* $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ *defined by*

$$V_{\hat{G}} = \{1, 2, 3, 4, 5, 6, 7\}$$

$$E_{\hat{G}} = \{12, 23, 14, 45, 16, 67\}.$$

This target graph gives

$$\Phi_S = \mathtt{Singleton}(V_{\hat{G}}, E_{\hat{G}}, 1, 0) = \begin{cases} 0 \quad 0 \rightleftharpoons 1-2, \ (r_1, r_2) \\ 2 \quad 0 \rightleftharpoons 3-4, \ (r_3, r_4) \\ 1 \quad 0 \rightleftharpoons 5-6, \ (r_5, r_6) \\ 6 \quad 0 \rightleftharpoons 7-8, \ (r_7, r_8) \\ 5 \quad 0 \rightleftharpoons 9-10, \ (r_9, r_{10}) \\ 10 \quad 0 \rightleftharpoons 11-12. \ (r_{11}, r_{12}) \end{cases}$$

This is the `Singleton` *rule set. Figure 17 illustrates one possible trajectory for assembly*

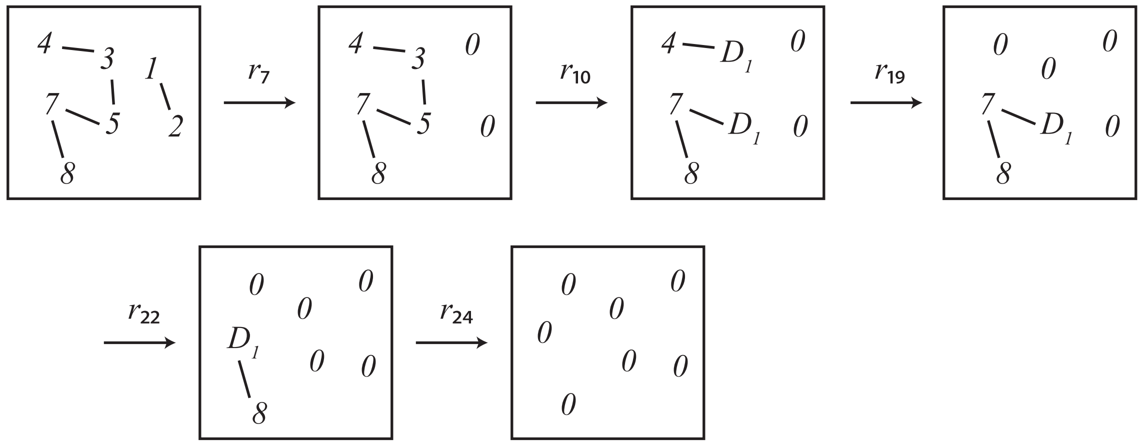

Figure 18: An example disassembly of a system in deadlock via the non-reversible `Singleton` process.

of $\hat{G}$. Notice that r_3 or r_7 could also have been the last rule, so all of r_3, r_5, and r_7 are completing rules.

The complete non-reversible `Singleton` *rule set is then given by*

$$
\Phi_N = \begin{cases}
0 \quad 0 \rightharpoonup 1-2, \ (r_1) \\
2 \quad 0 \rightharpoonup 3-4, \ (r_2) \\
1 \quad 0 \rightharpoonup 5-6, \ (r_3) \\
6 \quad 0 \rightharpoonup 7-8, \ (r_4) \\
5 \quad 0 \rightharpoonup 9-10, \ (r_5) \\
10 \quad 0 \rightharpoonup 11-12, \ (r_6) \\
1-2 \rightharpoonup 0 \quad 0, \ (r_7) \\
1-3 \rightharpoonup 0 \quad D_1, \ (r_8) \\
2-9 \rightharpoonup 0 \quad D_2, \ (r_9) \\
3-5 \rightharpoonup D_1 \quad D_1, \ (r_{10}) \\
5-6 \rightharpoonup D_1 \quad 0, \ (r_{11}) \\
5-7 \rightharpoonup D_1 \quad D_1, \ (r_{12}) \\
6-9 \rightharpoonup 0 \quad D_2, \ (r_{13}) \\
9-10 \rightharpoonup 0 \quad D_2, \ (r_{14}) \\
2-D_1 \rightharpoonup 0 \quad 0, \ (r_{15}) \\
2-D_2 \rightharpoonup 0 \quad D_1, \ (r_{16}) \\
3-D_1 \rightharpoonup D_1 \quad 0, \ (r_{17}) \\
3-D_2 \rightharpoonup D_1 \quad D_1, \ (r_{18}) \\
4-D_1 \rightharpoonup 0 \quad 0, \ (r_{19}) \\
6-D_1 \rightharpoonup 0 \quad 0, \ (r_{20}) \\
6-D_2 \rightharpoonup 0 \quad D_1, \ (r_{21}) \\
7-D_1 \rightharpoonup D_1 \quad 0, \ (r_{22}) \\
7-D_2 \rightharpoonup D_1 \quad D_1, \ (r_{23}) \\
8-D_1 \rightharpoonup 0 \quad 0, \ (r_{24})
\end{cases}
$$

where we have omitted the rules that are never applicable. Notice that for this graph, two D nodes never share an edge. Figure 18 illustrates how the added deconstructive rules can disassemble a structure in deadlock. Notice that one of the subgraphs would only have complements of completing rules applicable to it under the standard `Singleton` *process. The D nodes act as an infection that spreads through the assembly, disintegrating it in the process.* §

The following results are straightforward to verify.

Lemma 3.9.1 *For the non-reversible* `Singleton` *process, $\mathcal{Y}_{\hat{G}}(G_t)$ is nondecreasing in t.*

Lemma 3.9.2 *Suppose that $\mathcal{Y}_{\hat{G}}(G_t) < \lfloor N/|V_{\hat{G}}| \rfloor$, then there exists a length of time T and a probability $p > 0$ such that $\mathbf{Pr}\left[\mathcal{Y}_{\hat{G}}(G_{t+T}) > \mathcal{Y}_{\hat{G}}(G_t)\right] = p$.*

Theorem 3.9.1 *For the non-reversible* `Singleton` *process, $G_t \to \mathcal{G}_V^{\hat{G}}$ almost surely.*

3.10 The `MakeTree` algorithm

The `MakeTree` algorithm, presented in [1], shares some similarities with the algorithms developed in this paper. We review the algorithm here for the sake of completeness and to clearly delineate the differences. Like our algorithms, `MakeTree` generates binary constructive rules for any connected acyclic target. However, the algorithm is intended for the case of infinitely many parts and can exhibit deadlock when the number of parts is finite. Here, we consider the performance of the `MakeTree` algorithm under random pairwise selection dynamics. Note that we will add in the complementary rules so that we get a reversible version of `MakeTree`. Other than this, the differences are purely notational.

Like the other algorithms, `MakeTree` is a recursion that generates Φ from a target graph $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$.

We generate Φ using `MakeTree`$(V_{\hat{G}}, E_{\hat{G}}, 0)$. Note that line 4 can be implemented in more than one way, so different implementations of `MakeTree` can give different rule sets.

Algorithm 3 `MakeTree`(V, E, s)

if $|V| = 1$ **then**
    return $(\emptyset, \{(V, 0)\}, s)$
else
    choose $xy \in E$
    let (V_1, E_1) be the component of $(V, E - xy)$ containing x
    let (V_2, E_2) be the component of $(V, E - xy)$ containing y
    $(\Phi_1, l_1, s) \leftarrow$ `MakeTree`(V_1, E_1, s)
    $(\Phi_2, l_2, s) \leftarrow$ `MakeTree`(V_2, E_2, s)
    $\Phi \leftarrow \Phi_1 \cup \Phi_2 \cup \{l_1(x) \quad l_2(y) \rightleftharpoons (s+1) - (s+2)\}$
    $l \leftarrow (l_1 - \{(x, l_1(x))\}) \cup (l_2 - \{(y, l_2(y))\}) \cup \{(x, s+1), (y, s+2)\}$
    return $(\Phi, l, s+2)$
end if

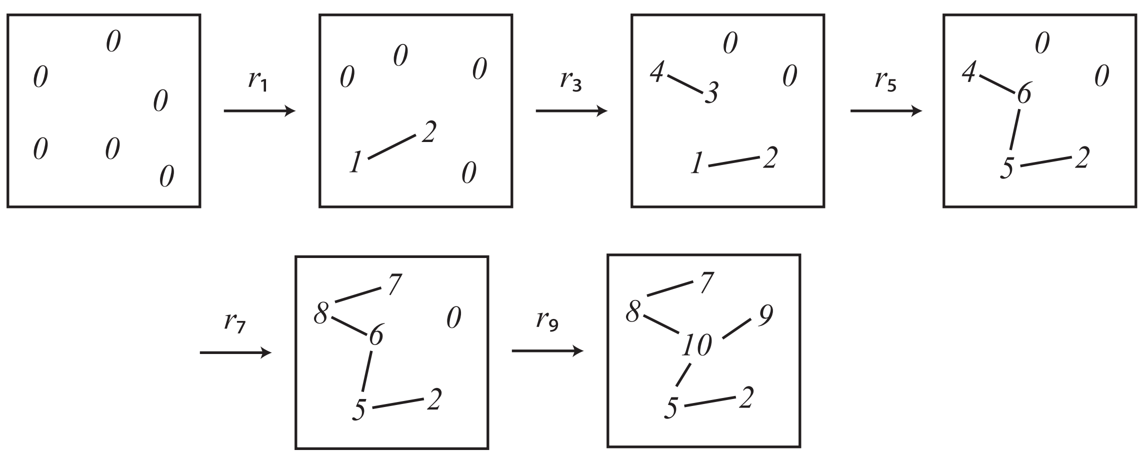

Figure 19: `Maketree` is not guaranteed to give rule sets with unique completing rules.

Example 3.10.1 (Lack of a completing rule) *While `MakeTree` will often produce rule sets with unique completing rules, we provide a counterexample here. Consider the following* $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ *defined by*

$$V_{\hat{G}} = \{1, 2, 3, 4, 5, 6\}$$

$$E_{\hat{G}} = \{12, 13, 34, 15, 56\}.$$

One possible rule set produced by `MakeTree`$(V_{\hat{G}}, E_{\hat{G}}, 0)$ *is*

$$\Phi = \begin{cases} 0 \quad 0 \rightleftharpoons 1-2, \ (r_1, r_2) \\ 0 \quad 0 \rightleftharpoons 3-4, \ (r_3, r_4) \\ 1 \quad 3 \rightleftharpoons 5-6, \ (r_5, r_6) \\ 0 \quad 4 \rightleftharpoons 7-8, \ (r_7, r_8) \\ 0 \quad 6 \rightleftharpoons 9-10. \ (r_9, r_{10}) \end{cases}$$

Figure 19 gives an example trajectory for this rule set that builds a target assembly. We could have reversed the order of r_7 *and* r_9*, indicating that this rule set does not give a unique completing rule.* §

Example 3.10.1 provides a counterexample to establish that `MakeTree` does not behave like `Lynchpin` in general. While we were able to impose reversibility on the `MakeTree` process by augmenting the rule set with complements, it is clear from the example that `MakeTree` does not satisfy the constraint of recoverable states. Nevertheless, simulations are suggestive that the reversible `MakeTree` process has an associated stochastic stability guarantee that is similar to that corresponding to the `Singleton` algorithm. This conjecture is unproven, however.

CHAPTER 4

SIGNALING GAMES

Signaling games[1] model communication between distributed, self-interested agents. Recent results for non-atomic signaling games establish the non-negligible possibility of convergence, under replicator dynamics, to states of unbounded efficiency loss. The effort to demonstrate that this is merely an artifact of the model has spawned alternatives that achieve maximum efficiency. Motivated by the empirical phenomenon of linguistic drift, we study the atomic signaling game under stochastic evolutionary dynamics. Our model does not converge in the manner of previously considered models, instead visiting all states infinitely often in the long-run. We analyze the long-run distribution of states and show that, in the small noise limit, its support is limited to the efficient communication systems. We find that this behavior is insensitive to the particular choice of evolutionary dynamic, a fact that is intuitively captured by the game's potential function corresponding to average fitness. Consequently, the model supports conclusions similar to those found in the literature on language competition. That is, we expect monomorphic language states to eventually predominate. Homophily has been identified as a feature that potentially stabilizes diverse linguistic communities. We find that incorporating homophily in our stochastic model gives mixed results. While the monomorphic prediction holds in the small noise limit, diversity can persist at higher noise levels or as a metastable phenomenon.

4.1 Introduction

Biological systems at many different scales depend on reliable and efficient signaling. Mathematical modeling of signaling may provide insights into conditions conducive to the emergence of communication in biological [54] and non-biological [55], [56] settings. A key problem is that of coordination. That is, how do systems coordinate on consistent

[1]These results appear in [51], [52], and [53].

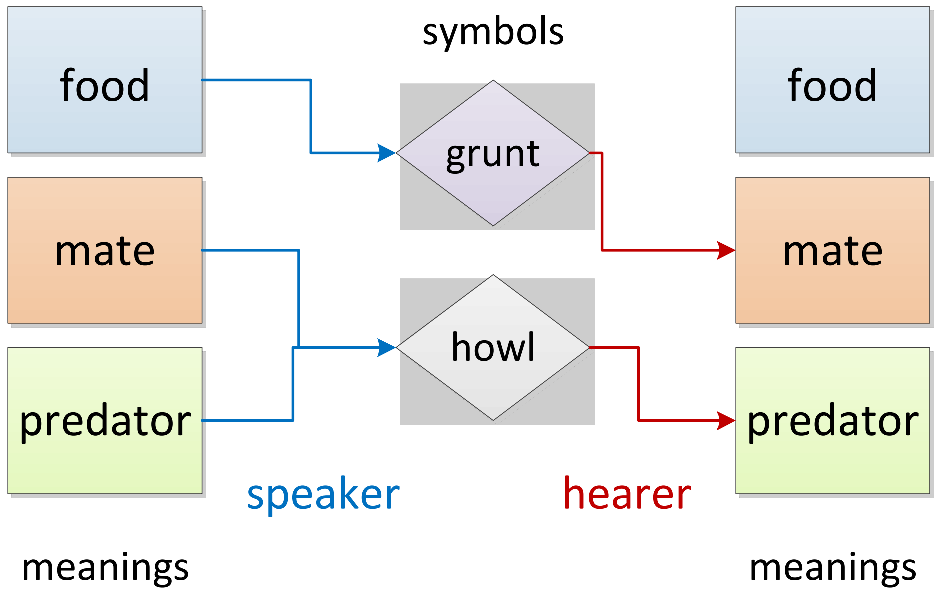

Figure 20: An illustration of the linguistic coherence achieved between a speaker and hearer.

communication protocols without the benefit of a centralized coordinating entity?

Signaling games [57] model the coordination problem in distributed communication. Researchers began studying these games in a biological context more recently [58], [59]. The strategies available to the players in a signaling game are pairs of mappings. A speaking strategy is a mapping from the set of objects to the set of symbols, while a hearing strategy is a mapping from the set of symbols back to the set of objects. A communication event involves two players and an object. The player assigned the role of speaker produces the symbol that her speaking strategy associates with the object. The player assigned the role of hearer then summons the object her hearing strategy associates with this symbol. If the final object object agrees with the original one then the event is successful, otherwise, it is not.

A fundamental question in signaling games is identification of conditions conducive to distributed learning of efficient communication systems. In the continuum agent, or, "non-atomic" setting it has been shown that the replicator dynamics can converge to neutrally stable states that do not maximize communicative efficiency from a set of initial conditions with positive measure [3], [4]. We nonetheless show that at least a binary communication system is almost always realized. Selection-mutation dynamics have been suggested [60] as an alternative to explain away the inefficiency. The system is analyzed for the special case of binary signaling games. The “mass action” perspective of non-atomic signaling games is taken up for analytical convenience. The more realistic discrete agent, or, “atomic” model is approximated by the non-atomic model over finite time horizons for sufficiently large populations [28]. Characterization of states favored by selection in the frequency dependent Moran process [6] has been carried out for the non-atomic signaling game. Essentially, this analysis says that the efficient states are the most robust to a one-time stochastic shock in the form of the introduction of a single mutant agent. In this paper we study the long-run behavior of stochastic evolutionary dynamics in the non-atomic signaling game. That is, we examine the behavior of the system under persistent random excitations. Although our analysis will concentrate on the situation where these shocks occur with arbitrarily low probability, the outcomes in such a scenario can still be qualitatively different from models that consider random excitations in isolation or ignore them altogether [61].

As a starting point we show that the non-atomic signaling game is a potential game [12]. Some algorithms exist (see for instance [25]) that are equipped with substantive performance guarantees for all or some of the potential games. We focus on evolutionary dynamics that closely mimic the replicator dynamics. If more clever dynamics are utilized the problem becomes quite trivial, which is good news for the proactive engineering of communicating agents. In essence, potential games reduce the individual, myopic optimization activities of distributed agents to centralized optimization of the so-called potential function. Since the potential function of the atomic signaling game is proportional to average fitness, that dynamics resembling natural selection[2] with random mutations achieve maximum average fitness is intuitively reasonable. We propose a number of evolutionary

[2]Our results can also be interpreted in the context of cultural evolution, but we emphasize the biological interpretation first and foremost.

dynamics achieving just that. In our models, agents mostly coordinate on a single language, but the form of that language will change over time consistent with the empirically observed phenomenon of linguistic drift [62].

While the first part of this paper is concerned with the problem of justifying the emergence of efficient signaling by evolutionary processes, the second part instead attempts to explain the failure to observe such perfect coordination in natural language. Some have argue, using models of language competition that are mathematically similar to our own, that this state of affairs is transient [63]. More sophisticated models incorporating, for instance, a spatial dimension [64] have added nuance to the problem. The very same neutrally stable states that precluded efficient communication in the non-atomic signaling game have also been brought to bear on this problem [65] as a mechanism for initiating diversity, although exogenous criteria like isolation would be needed to sustain it. In recent years, homophily, the tendency of individuals to associate with similar others, has been suggested as a counterweight to the otherwise decisive tendency towards homogeneity in communication protocols [7]. This approach is consistent with the idea that languange can function as an in-group marker [66]. We examine whether incorporating such features into dynamics in atomic signaling games succeeds at engendering diverse linguistic communities that achieve high efficiency internally, but not externally. To this end, we attain mixed results. We are able to show for a restricted set of parameters that as the frequency of randomizing behavior is taken to zero the long-run behavior is no different from our original models. On the other hand, when randomizing activity is more frequent we observe encouraging results in simulations. Also, even when the crucial randomizing activity is extremely rare, diversity may persist as a metastable phenomenon.

4.1.1 Signaling games

Formally, there are $m \geq 2$ objects and $n \geq 2$ symbols. A speaking strategy is represented by an $m \times n$ binary, row stochastic matrix[3] P. If $P_{ij} = 1$ then the speaking strategy associates object i with symbol j. Similarly, a hearing strategy is an $n \times m$ binary, row-stochastic matrix Q. If $Q_{ij} = 1$ then the hearing strategy associates symbol i with object j. Thus, assuming uniform probability over objects, the symmetric payoff of players utilizing P and Q is

$$\sum_i \sum_j P_{ij} Q_{ji} = \text{trace}(PQ).$$

We call a joint speaking and hearing strategy (P, Q) a *language* and use $\mathcal{L}_{m,n}$ to refer to the set of all such pairs. In this chapter, we will study the both the atomic and non-atomic signaling game. The difference is whether or not there is a finite set of players, with each one selecting her own speaking and hearing strategies. We first review the non-atomic version of the game.

4.2 The non-atomic signaling game

The set $\mathcal{L}_{m,n}$ has cardinality $m^n n^m$. Confer any ordering on the elements of $\mathcal{L}_{m,n}$ so that the $m^n n^m$-dimensional simplex

$$S_{m,n} = \{\mathbf{x} \in \mathbb{R}^{m^n n^m} : \sum_i x_i = 1, x_i \geq 0 \quad \forall i\}$$

gives the possible distributions of a single population over the set of languages. Let (P^k, Q^k) be the kth language in the ordering. Then the fitness of a player utilizing (P^k, Q^k) in a population state $\mathbf{x}$ is

$$f_k(\mathbf{x}) = \sum_{i=1}^{m^n n^m} x_i \left(\text{trace}(P^k Q^i) + \text{trace}(P^i Q^k)\right).$$

In other words, the fitness of a player is the expected payoff from uniform random matching with the population, with equal weighting given to speaking and hearing. Analyses of this

[3] A binary, row-stochastic matrix is a matrix that has one element per row that is equal to one and all other elements equal to zero.

game have concentrated on the replicator dynamics

$$\dot{x}_i = x_i \left(f_i(x) - \sum_j x_j f_j(x) \right).$$

It has been shown [3], [4],[5] that the replicator dynamics do not almost always converge to states maximizing average fitness

$$W(\mathbf{x}) = \sum_{i=1}^{m^n n^m} x_i f_i(\mathbf{x}).$$

A population state $\mathbf{x}$ is a *Neutrally Stable State* (NSS) if

$$\mathbf{x}' f(\mathbf{x}) \geq \mathbf{y}' f(\mathbf{x}) \quad \forall \mathbf{y} \in X,$$

and if $\mathbf{x}' f(\mathbf{x}) = \mathbf{y}' f(\mathbf{x})$ then $\mathbf{x}' f(\mathbf{y}) \geq \mathbf{y}' f(\mathbf{y})$. The set of attractors of the language game under replicator dynamics are called the NSS. We may converge on NSS with average fitness of four for any value of m or n [60]. This fact is particularly unsettling taking into account that the maximum average fitness is $2\min\{m, n\}$. We find that this is the worst-case.

Theorem 4.2.1 *If $\boldsymbol{x}$ is an NSS then $\sum_i x_i f_i(\boldsymbol{x}) \geq 4$ and the bound is tight for all $m, n \geq 2$.*

The proof can be sketched as follows. If a counterexample exists, i.e. an NSS with a smaller average fitness, then it can be shown by induction that such an NSS exists for the case of $m = n = 2$. It is then simple to show there is no such NSS when $m = n = 2$.

4.2.1 Proof of Theorem 4.2.1

It is straightforward to construct population states satisfying the bound with equality for any m, n. To see this, note that we can associate any population state $\mathbf{x}$ with the average speaker and hearer,

$$(\overline{P}, \overline{Q}) = (\sum_j x_j P_j, \sum_j x_j Q_j).$$

This way, the set of average languages is simply the product of the set of $m \times n$ row-stochastic matrices and the set of $n \times m$ row-stochastic matrices. In [60], NSS achieving average fitness of four are described for the case of $m = n$. We trivially extend their example to the case of general m, n. Consider $(\overline{P}, \overline{Q})$ given by

$$\left(\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \mu_1 & \cdots & \mu_{m-1} \end{bmatrix} \right),$$

where $0 \leq \lambda_i, \mu_j \leq 1$ for all i, j. Clearly $\text{trace}(\overline{P}\overline{Q}) = \sum_i \lambda_i + \sum_j \mu_j = 2$, implying an average fitness of four. That the corresponding population state is an NSS follows from the following lemma, which is merely a combination of Theorem 1 and 2 in [6].

Lemma 4.2.1 *An average language (P, Q) correponds to an NSS if and only if it satisfies the following four conditions:*

1.

$$P \in \operatorname*{argmax}_{\hat{P} \text{ row-stochastic}} \text{trace}(\hat{P}Q),$$

2.

$$Q \in \operatorname*{argmax}_{\hat{Q} \text{ row-stochastic}} \text{trace}(P\hat{Q}),$$

3. at least one of P, Q has no zero-column, and

4. neither P nor Q has a column with multiple maximal elements in $(0, 1)$.

The first two conditions are neccesary and sufficient for (P, Q) to be a Nash equilibrium and the second two conditions are neccesarry and sufficient for a Nash equilibrium to be an NSS in this setting. In order to prove our theorem, we must show that no NSS can achieve average fitness strictly less than four.

Suppose there exists $(\overline{P}, \overline{Q})$ that is an NSS and satisfies $\text{trace}(\overline{P}\overline{Q}) < 2$, so that it corresponds to an average fitness less than four. It follows that there are $m - 1$ rows of $\overline{P}$ that contribute less than one to trace($\overline{P}\,\overline{Q}$). That is, the set

$$\mathcal{R} = \{i : \sum_j \overline{P}_{ij}\overline{Q}_{ji} < 1\},$$

has cardinality at least $m-1$. The next lemma shows that it suffices to consider $\overline{P}$ for which no two of these rows are the same standard basis vector.

Lemma 4.2.2 *Suppose $m > 2, n \geq 2$ and (P, Q) is the average language corresponding to an NSS. Further suppose that there exist i_1, i_2 such that $P_{i_1 k} = P_{i_2 k} = 1$ for some k. Then there exists $(\hat{P}, \hat{Q})$ that corresponds to an NSS for the reduced game with dimensions $(m-1), n$ and also satisfies* $\mathrm{trace}(PQ) = \mathrm{trace}(\hat{P}\hat{Q})$.

Proof: Given (P, Q) and assuming without loss of generality that $i_1 = 1$ and $i_2 = 2$ we can construct $(\hat{P}, \hat{Q})$ as

$$\hat{P}_{ij} = \begin{cases} P_{1j}, & i = 1 \\ P_{(i+1)j}, & i \neq 1 \end{cases},$$

so that $\hat{P}$ consolidates the identical first two rows of P. For Q, we consolidate by combining the first two columns (i_1 and i_2), so that

$$\hat{Q}_{ij} = \begin{cases} Q_{i1} + Q_{i2}, & j = 1 \\ Q_{i(j+1)}, & j \neq 1 \end{cases}.$$

In order to complete the proof of the lemma we must verify both that $(\hat{P}, \hat{Q})$ corresponds to an NSS of the reduced-order game and that $\mathrm{trace}(PQ) = \mathrm{trace}(\hat{P}\hat{Q})$. The latter is easily verified by expanding out,

$$\begin{aligned}
\mathrm{trace}(\hat{P}\hat{Q}) &= \sum_i \sum_j \hat{P}_{ij}\hat{Q}_{ji} = \sum_{i\neq 1} \sum_j P_{(i+1)j}Q_{j(i+1)} + \sum_j \hat{P}_{1j}\hat{Q}_{j1} \\
&= \sum_{i\neq 1} \sum_j P_{(i+1)j}Q_{j(i+1)} + \sum_j \hat{P}_{1j}(Q_{j1} + Q_{j2}) \\
&= \sum_{i\neq 1} \sum_j P_{(i+1)j}Q_{j(i+1)} + \sum_j P_{1j}Q_{j1} + \sum_j P_{2j}Q_{j2} \\
&= \mathrm{trace}(PQ),
\end{aligned}$$

as required.

To verify the NSS property we must show that the four conditions in Lemma 4.2.1 are satisfied. For the first condition, assume the contrary, i.e. there exists $\tilde{P}$ such that $\mathrm{trace}(\tilde{P}\hat{Q}) > \mathrm{trace}(\hat{P}\hat{Q})$. The matrix $\tilde{P}$ can be expanded into a speaker matrix for the $m \times n$ game given by

$$P^*_{ij} = \begin{cases} \tilde{P}_{1j}, & i \in \{1,2\} \\ \tilde{P}_{(i-1)j}, & i \notin \{1,2\} \end{cases}.$$

We show that the existence of P^* contradicts the supposition that (P, Q) is an NSS. This follows from computing

$$\begin{aligned} \mathrm{trace}(P^*Q) &= \sum_i \sum_j P^*_{ij} Q_{ji} \\ &= \sum_{i \notin \{1,2\}} \sum_j \tilde{P}_{(i-1)j} Q_{ji} + \sum_j \tilde{P}_{1j} Q_{j1} + \sum_j \tilde{P}_{1j} Q_{j2} \\ &= \sum_{i \notin \{1,2\}} \sum_j \tilde{P}_{(i-1)j} \hat{Q}_{j(i-1)} + \sum_j \tilde{P}_{1j} \hat{Q}_{j1} \\ &= \mathrm{trace}(\tilde{P}\hat{Q}) > \mathrm{trace}(\hat{P}\hat{Q}) = \mathrm{trace}(PQ), \end{aligned}$$

implying that (P, Q) does not satisfy the first condition of Lemma 4.2.1.

The second condition can be verified similarly. Assume the contrary, i.e. there exists $\tilde{Q}$ such that $\mathrm{trace}(\hat{P}\tilde{Q}) > \mathrm{trace}(\hat{P}\hat{Q})$. We define

$$Q^*_{ij} = \begin{cases} \frac{\tilde{Q}_{i1}}{2}, & j \in \{1,2\} \\ \tilde{Q}_{i(j-1)}, & j \notin \{1,2\} \end{cases},$$

so that we have

$$\begin{aligned}
\operatorname{trace}(PQ^*) &= \sum_i \sum_j P_{ij} Q^*_{ji} \\
&= \sum_{i \notin \{1,2\}} \sum_i P_{ij} \tilde{Q}_{j(i-1)} + \sum_j P_{1j} \frac{\tilde{Q}_{j1}}{2} + \sum_j P_{2j} \frac{\tilde{Q}_{j1}}{2} \\
&= \sum_{i \notin \{1,2\}} \sum_j \hat{P}_{(i-1)j} \tilde{Q}_{j(i-1)} + \sum_j \hat{P}_{1j} \tilde{Q}_{j1} \\
&= \operatorname{trace}(\hat{P}\tilde{Q}) > \operatorname{trace}(\hat{P}\hat{Q}) = \operatorname{trace}(PQ),
\end{aligned}$$

implying that (P, Q) does not satisfy the second condition of Lemma 4.2.1.

For the third condition of Lemma 4.2.1 first suppose P has no zero column. Consider the sum of the j'th column of $\hat{P}$. If $j = k$ then

$$\sum_i \hat{P}_{ik} = P_{1k} + \sum_{i \neq 1} P_{(i+1)k} = 1 + \sum_{i \neq 1} P_{(i+1)k} > 0,$$

as required. If $j \neq k$ then

$$\sum_i \hat{P}_{ij} = P_{1j} + \sum_{i \neq 1} P_{(i+1)j} = P_{1j} + P_{2j} + \sum_{i \neq 1} P_{(i+1)j} = \sum_i P_{ij} > 0,$$

as required. Instead suppose Q has no zero column. Aside from the first column, all the columns of $\hat{Q}$ match columns of Q and are hence non-zero. The first column sum is given by

$$\sum_i \hat{Q}_{i1} = \sum_i Q_{i1} + Q_{i2} > \sum_i Q_{i1} > 0,$$

as required.

Finally, we verify the fourth condition. First consider $\hat{P}$. The k'th column has

$$\max_i \hat{P}_{ik} = P_{1k} = 1,$$

by assumption. For any other column $j \neq k$ we have two cases. If $\max_i P_{ij} = 1$ then so does $\max_i \hat{P}_{ij}$. Alternatively if $\max_i P_{ij} \in (0, 1)$ then the cardinality of $\operatorname{argmax}_i \hat{P}_{ij}$ is still one because P_{1j}, the deleted element, is equal to zero. Lastly, consider the first column of $\hat{Q}$. The other columns are unchanged from Q. Suppose that $\max_i \hat{Q}_{i1} \in (0, 1)$. This requires that $\max_i Q_{i1} \in [0, 1)$ and $\max_i Q_{i2} \in [0, 1)$, with at most one of the quantities equal to zero. Assume without loss of generality that $\max_i Q_{i1} \in (0, 1)$. If $\max_i Q_{i2} = 0$ then $\hat{Q}_{i1} = Q_{i1}$ for all i and the condition is satisfied. Therefore, suppose $\max_i Q_{i2} \in (0, 1)$ as well. We claim that

$$\operatorname*{argmax}_i \hat{Q}_{i1} = \operatorname*{argmax}_i Q_{i1} = k = \operatorname*{argmax}_i Q_{i2},$$

which clearly implies the fourth condition. The first equality follows from the second and third equalities and the definition of $\hat{Q}$. The second and third equalities can be verified by supposing $\operatorname{argmax}_i Q_{i\hat{j}} = \hat{k} \neq k$ for some $\hat{j} \in \{1, 2\}$. We could then define

$$\tilde{P}_{ij} = \begin{cases} P_{ij}, & i \neq \hat{j} \\ 1, & i = \hat{j}, j = \hat{k} \\ 0, & i = \hat{j}, j \neq \hat{k} \end{cases},$$

so that

$$\begin{aligned} \operatorname{trace}(\tilde{P}Q) - \operatorname{trace}(PQ) &= \sum_j \tilde{P}_{\hat{j}j} Q_{j\hat{j}} - \sum_j P_{\hat{j}j} Q_{j\hat{j}} \\ &= Q_{\hat{k}\hat{j}} - Q_{k\hat{j}} > 0, \end{aligned}$$

which contradicts condition one of Lemma 4.2.1 for (P, Q). ■

As mentioned above, Lemma 4.2.2 allows us to assume that $\bar{P}$ contains no two rows that are the same standard basis vector. This is because we can apply Lemma 4.2.2 inductively until there are no more repeated standard basis vectors. If a counterexample to the theorem exists for the higher-dimensional game with the repeated basis vectors, then the existence of a counter-example is also implied for the lower-dimensional game sans the repeated basis vector. That is, so long as $m > 2$ and $n \geq 2$. It turns out that there are no such NSS when $m = 2$.

Lemma 4.2.3 *Suppose $m = 2, n \geq 2$ and P's rows are both the same standard basis vector. Then (P, Q) is not the average language corresponding to an NSS.*

Proof: Asssume that (P, Q) is an NSS. We can assume without loss of generality that

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

It follows that $\mathrm{trace}(PQ) = Q_{11} + Q_{12}$. We claim that

$$1 \in \operatorname*{argmax}_i Q_{ij}, \text{ for } j \in \{1, 2\}.$$

To see this, assume the contrary, i.e. that there exists $\hat{i} \neq 1, \hat{j} \in \{1, 2\}$ such that $Q_{\hat{i}\hat{j}} > Q_{1\hat{j}}$. Then construct

$$\tilde{P}_{ij} = \begin{cases} P_{ij}, & i \neq \hat{j} \\ 1, & i = \hat{j}, j = \hat{i} \\ 0, & i = \hat{j}, j \neq \hat{i} \end{cases},$$

and observe that

$$\mathrm{trace}(\tilde{P}Q) - \mathrm{trace}(PQ) = Q_{\hat{i}\hat{j}} - Q_{1\hat{j}} > 0,$$

which contradicts our assumption that (P, Q) is an NSS because the first condition of Lemma 4.2.1 is violated. Next, assume without loss of generality that $1 > Q_{11} \geq \frac{1}{2}$, implying $0 < Q_{12} \leq \frac{1}{2}$. The strict inequalities are implied by the third condition of Lemma 4.2.1 along with the preceding claim. The second row must also sum to one, so its maximum element must be at least $\frac{1}{2}$. If the maximum element is in the first column then it must be strictly less than Q_{11} due to the fourth condition of Lemma 4.2.1. It follows that

$$Q_{22} = 1 - Q_{21} > 1 - Q_{11} = Q_{12},$$

which contradicts our claim. If the maximum element of the second row is in the second column then $Q_{22} \geq \frac{1}{2} \geq Q_{12}$, which contradicts our claim since the fourth condition of Lemma 4.2.1 implies $Q_{22} \neq Q_{12}$. ■.

The two preceding lemmas allow us to assume that no two rows in $\mathcal{R}$ are the same standard basis vector. It follows that there exists $\hat{i} \in \mathcal{R}$ and $\hat{j}$ such that

$$\{\hat{i}\} = \operatorname*{argmax}_i \overline{P}_{i\hat{j}}.$$

In other words, some element in one of the rows in $\mathcal{R}$ is the unique maximum element in its column. To see this, assume the contrary. That is, each column of P either has its unique maximum element in the row not included in $\mathcal{R}$, or has multiple maximum elements. If a column has multiple maximum elements then those elements must all be equal to one by the fourth condition of Lemma 4.2.1. Since no two rows of $\bar{P}$ are the same standard basis vector this implies that the row not in $\mathcal{R}$ contains the other one, with its other elements being zero. Remaining columns cannot have multiple maximum elements or their maximum elements in the row not in $\mathcal{R}$, a contradiction. If all column maxima are unique, then one of the rows in $\mathcal{R}$ must contain such a maximum by row-stochasticity.

The existence of $\hat{\imath}$, $\hat{\jmath}$ implies that $\bar{Q}_{\hat{\jmath}\hat{\imath}} = 1$. If it did not then we could construct

$$\tilde{Q}_{ij} = \begin{cases} \bar{Q}_{ij}, & i \neq \hat{\jmath} \\ 1, & i = \hat{\jmath}, j = \hat{\imath} \\ 0, & i = \hat{\jmath}, j \neq \hat{\imath} \end{cases},$$

so that

$$\operatorname{trace}(\bar{P}\tilde{Q}) - \operatorname{trace}(\bar{P}\bar{Q}) = 1 - \bar{Q}_{\hat{\jmath}\hat{\imath}} > 0,$$

which contradicts the second condition of Lemma 4.2.1. We conclude the proof of Theorem 4.2.1 by demonstrating that $\hat{\imath} \notin \mathcal{R}$, a contradiction. Otherwise we could construct

$$\tilde{P}_{ij} = \begin{cases} \bar{P}_{ij}, & i \neq \hat{\imath} \\ 1, & i = \hat{\imath}, j = \hat{\imath} \\ 0, & i = \hat{\imath}, j \neq \hat{\jmath} \end{cases},$$

so that

$$\operatorname{trace}(\tilde{P}\bar{Q}) - \operatorname{trace}(\bar{P}\bar{Q}) = 1 - \sum_j \bar{P}_{\hat{\imath}j}\bar{Q}_{j\hat{\imath}} > 0,$$

which contradicts the second condition of Lemma 4.2.1, where the inequality is simply the definition of membership in $\mathcal{R}$. ■

Theorem 4.2.1 establishes that the replicator dynamics almost always leads to at least a binary communication system. Recall that a communication system wherein two symbols are correctly conveyed in the average communication event achieves a fitness of four because we count both speaking and hearing in our accounting of fitness. The language game has Nash equilibria achieving fitness as low as two. These equilibria are fixed points of the replicator dynamics, but they do not attract a set of initial conditions with positive measure. From an information theoretic perspective, these two situations are vastly different since efficient utilization of two symbols corresponds to one bit of information, while one symbol corresponds to zero bits of information.

In order to reconcile this overall optimality gap with the intuitive notion that evolution leads to efficient signaling, a number of alternative models have been proposed. Mutation-selection dynamics, a perturbation of the replicator dynamics have been studied in the binary case [60]. These dynamics add a "mutation" term to the replicator dynamics, intended to capture the effect of random mutations. In non-biological contexts this term can be interpreted as experimentation. Any type is equally likely to mutate into any other type, with emphasis on the limiting behavior as this rate goes to zero. In the next section we will study a somewhat similar dynamic. However, we instead concentrate on the finite-population, or "atomic" game. The motivation for considering atomic agents is that it enables us to analyze the long-run behavior of stochastic evolutionary dynamics. A common justification for studying mass-action heuristics like the non-atomic signaling game is the fact that, over short time horizons, these models approximate stochastic evolutionary dynamics with sufficiently many atomic agents [28].

4.3 The atomic signaling game

We consider N agents, each utilizing a particular language. Let $(\mathbf{P}, \mathbf{Q}) \in \mathcal{L}_{m,n}^N$ be an vector of N languages, one for each agent. Then the fitness of agent $i \in \{1, ..., N\}$ is

$$f_i(\mathbf{P}, \mathbf{Q}) = \text{trace}(P_i \frac{1}{N-1} \sum_{j \neq i} Q_j) + \text{trace}(\frac{1}{N-1} \sum_{j \neq i} P_j Q_i),$$

analogous to the infinite-population model. The frequency dependent Moran process has been analyzed for this game [6], establishing that evolution tends towards efficient states in the limit of weak selection. We instead seek to characterize the long-run behavior of this game under stochastic evolutionary dynamics. In particular, we suppose that at each discrete time instant t, a player is randomly selected for a strategy revision oppurtunity according to a family of probability distributions

$$F : \mathcal{L}_{m,n}^{N} \to \mathrm{int}(S_N),$$

where $\mathrm{int}(\cdot)$ is the interior of a set. That is, given $(\mathbf{P}[t-1], \mathbf{Q}[t-1])$, we select a player i according to $F(\mathbf{P}[t-1], \mathbf{Q}[t-1]) > 0$. The selected player updates her strategy as

$$(P_i[t], Q_i[t]) = \begin{cases} (P_{\hat{k}}, Q_{\hat{k}}), & \text{with probability } 1-\epsilon \\ \mathrm{rand}(\mathcal{L}_{m,n}), & \text{with probability } \epsilon \end{cases} \tag{1}$$

where $\hat{k} \in \mathrm{argmax}_k\, f_k(\mathbf{P}[t-1], \mathbf{Q}[t-1])$ and $\mathrm{rand}(\cdot)$ indicates the outcome of uniform random sampling from the given set. The parameter $\epsilon > 0$ indicates the mutation probability. We choose $\hat{k}$ from $\mathrm{argmax}_k\, f_k(\mathbf{P}[t-1], \mathbf{Q}[t-1])$ at random uniformly if the set contains multiple elements. All other players leave their strategies unchanged, i.e.

$$(P_j[t], Q_j[t]) = (P_j[t-1], Q_j[t-1]) \quad \forall j \neq i.$$

This dynamic is of course not exactly like the replicator dynamics. In particular, as opposed to reproduction in proportion to fitness, reproductive opportunities are afforded to only the most fit players. On the other hand, the dynamic preserves the feature of imitation, that is, unused strategies are not subsequently taken up apart from via rare mutations. We will revisit the form of the dynamics in Section 4.4 and suggest some variations that are more realistic. Since the analysis of those models is a straightforward extension of the results for the present model, we introduce the relevant analytical concepts here.

4.3.1 Potential games

A game is *a potential game* [12] if there exists a function $\Phi : \mathcal{A} \to \mathbb{R}$ (the domain being the set of joint strategies) such that for any player i, any joint strategy $\mathbf{s}$, and any strategy s of player i we have

$$f_i(s, \mathbf{s}_{-i}) - f_i(\mathbf{s}) = \Phi(s, \mathbf{s}_{-i}) - \Phi(\mathbf{s}),$$

where $\mathbf{s}_{-i}$ is the vector of strategiess for players other than i. The implication of this definition is that individual optimizing activity is tantamount to optimization of the potential function Φ.

Theorem 4.3.1 *The finite-population language game is a potential game*[4] *with potential function* $\Phi \equiv \frac{1}{2}\sum_{i=1}^{N} f_i$.

Proof: Let $(\mathbf{P}, \mathbf{Q})$ and $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ differ only in the language of player $\hat{i}$. Then we have $\Phi(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) - \Phi(\mathbf{P}, \mathbf{Q})$

$$\begin{aligned}
&= \frac{1}{2}\sum_i f_i(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) - \frac{1}{2}\sum_i f_i(\mathbf{P}, \mathbf{Q}) \\
&= \frac{1}{2}\sum_i \left(\mathrm{trace}(\hat{P}_i \frac{1}{N-1}\sum_{j\neq i} \hat{Q}_j) + \mathrm{trace}(\frac{1}{N-1}\sum_{j\neq i} \hat{P}_j \hat{Q}_i)\right) \\
&\quad - \frac{1}{2}\sum_i \left(\mathrm{trace}(P_i \frac{1}{N-1}\sum_{j\neq i} Q_j) + \mathrm{trace}(\frac{1}{N-1}\sum_{j\neq i} P_j Q_i)\right) \\
&= \frac{1}{N-1}\sum_{i\neq\hat{i}}\sum_{j\notin\{i,\hat{i}\}} \mathrm{trace}(P_i Q_j + P_j Q_i) + \frac{1}{N-1}\sum_{j\neq\hat{i}} \mathrm{trace}(\hat{P}_{\hat{i}} \hat{Q}_j + \hat{P}_j \hat{Q}_{\hat{i}}) \\
&\quad - \frac{1}{2(N-1)}\sum_{i\neq\hat{i}}\sum_{j\notin\{i,\hat{i}\}} \mathrm{trace}(P_i Q_j + P_j Q_i) - \frac{1}{N-1}\sum_{j\neq\hat{i}} \mathrm{trace}(P_{\hat{i}} Q_j + P_j Q_{\hat{i}}) \\
&= \mathrm{trace}(\hat{P}_{\hat{i}} \frac{1}{N-1}\sum_{j\neq\hat{i}} \hat{Q}_j) + \mathrm{trace}(\frac{1}{N-1}\sum_{j\neq\hat{i}} \hat{P}_j \hat{Q}_{\hat{i}}) \\
&\quad - \mathrm{trace}(P_{\hat{i}} \frac{1}{N-1}\sum_{j\neq\hat{i}} Q_j) + \mathrm{trace}(\frac{1}{N-1}\sum_{j\neq\hat{i}} P_j Q_{\hat{i}}) \\
&= f_{\hat{i}}(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) - f_{\hat{i}}(\mathbf{P}, \mathbf{Q}),
\end{aligned}$$

[4] Analogous definitions exist for non-atomic games, but are not taken up here, see [26].

as required. ■

Since the potential function is proportional to average fitness, it is somewhat unsuprising that many stochastic evolutionary dynamics tend to maximize average fitness. Indeed, from the perspective of distributed algorithm design problems of this form are well studied and generic procedures with strong performance guarantees exist. For instance, under logit dynamics players spend almost all of their time at maximizers of the potential function over the long run as a temperature parameter is taken to zero [25]. However, logit bears little resemblance to the replicator dynamics studied in the non-atomic signaling game.

Many variations on logit have been suggested, motivated by concerns such as information and actuation constraints in engineering applications [67] or behavioral tendencies and rates of convergence [68]. Our work is novel in its insistence on replicator-like dynamics, although see [69]. The dynamics we suggest are analyzed only for the atomic signaling game. Characterizing the equilibrium selection properties and rates of convergence of such algorithms more generally is a future direction.

Let $\mathcal{P}^{\epsilon}_{m,n}$ be the transition matrix corresponding to the Markov chain described above. It is straightforward to see that this process is irreducible and aperiodic, thus admitting a unique stationary distribution. We seek to make statements about this distribution that will be valid when ϵ is small.

Let $\mathcal{T}_{R_i}$ be the set of resistance trees rooted at the recurrent class R_i We will repeatedly make use of the following simple and useful result.

Lemma 4.3.1 *Let $\mathcal{M}^{\epsilon}$ be a regular perturbed Markov process and let $R_1, ..., R_K$ be the recurrent classes of $\mathcal{M}^0$, then for any recurrent class R_i we have*

$$\gamma(R_i) \geq \sum_{j \neq i} \min_{k \neq j} r_{jk} \equiv \lambda(R_i).$$

Proof:

$$\gamma(R_i) = \min_{T \in \mathcal{T}_{R_i}} \sum_{(R_j, R_k) \in T} r_{jk} \geq \min_{T \in \mathcal{T}_{R_i}} \sum_{(R_j, R_k) \in T} \min_{l \neq j} r_{jl} = \sum_{j \neq i} \min_{l \neq j} r_{jl},$$

where the last equality follows from the fact that the directed edges of all spanning trees have the same set of source vertices. ■

In the next section we characterize the reccurent communication classes of $\mathcal{P}^0_{m,n}$

4.3.2 Properties of $\mathcal{P}^0_{m,n}$

The recurrent communication classes of $\mathcal{P}^0_{m,n}$ are singletons, or, abosrbing states. In particular, they are the states in which all players use a single language so that

$$(\mathbf{P}, \mathbf{Q}) = ((P, Q), (P, Q), ..., (P, Q)),$$

for some $(P, Q) \in \mathcal{L}_{m,n}$. We call these states monomorphic. If a language (P, Q) satisfies $\mathrm{trace}(PQ) = \min\{m, n\}$, the maximum, then we call that language *aligned*. We call the associated monomorphic state *optimal*. Let $\mathcal{O}$ refer to the set of optimal states. The set of optimal states maximize our efficiency measure, the average fitness

$$W(\mathbf{P}, \mathbf{Q}) = \frac{1}{N} \sum_{k=1}^{N} f_k(\mathbf{P}, \mathbf{Q}).$$

In the next section, we identify the stochastically stable states of $\mathcal{P}^\epsilon_{m,n}$, first for the case where the number of symbols match ($m = n$). The case where they do not ($m \neq n$), is taken up in Section 4.3.4.

4.3.3 Efficiency of $\mathcal{P}^\epsilon_{m,n}$ when $m = n$

For sufficiently small ϵ, $\mathcal{P}^\epsilon_{m,m}$ spends an arbitrarily large proportion of its time in optimal states.

Theorem 4.3.2 *A state of $\mathcal{P}^\epsilon_{m,m}$ is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states .*

The remainder of this section will develop the proof of the above theorem. We will need to state and prove several lemmas along the way. In the next section we will address the case of $m \neq n$.

In order to make use of Lemma 4.3.1 for $\mathcal{P}^\epsilon_{m,m}$, we need to find $\min_{k \neq j} r_{jk}$ for each recurrent class R_j. We first consider optimal states.

Lemma 4.3.2 *For any $R_s \in \mathcal{O}$ we have* $\min_{k \neq s} r_{sk} = \lceil N/2 \rceil$ *and for every $R_t \in \mathcal{O} - R_s$ we have $t \in$* $\operatorname{argmin}_{k \neq s} r_{sk}$.

Proof: Suppose that in state $(\mathbf{P}, \mathbf{Q})$ we have $k_1 \geq \lfloor N/2 \rfloor + 1$ players using an aligned language $(\hat{P}, \hat{Q})$, implying that there are $K \leq \lceil N/2 \rceil$ distinct languages in the population. Let $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}) \in \mathcal{L}^K_{m,n}$ be the vector of distinct languages, with $(\tilde{P}_1, \tilde{Q}_1) = (\hat{P}, \hat{Q})$. Also, let $k_i, i \in \{2, ..., K\}$ indicate the number of players using $(\tilde{P}_i, \tilde{Q}_i)$ so that $\sum_{k=1}^{K} k_i = N$. We claim that from the state $(\mathbf{P}, \mathbf{Q})$, the unperturbed process $\mathcal{P}^0_{m,m}$ returns to the optimal state

$$(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) = \left((\hat{P}, \hat{Q}), (\hat{P}, \hat{Q}), ..., (\hat{P}, \hat{Q})\right),$$

with probability one. We need to show that in (1), we have $(P_{\hat{k}}, Q_{\hat{k}}) = (\hat{P}, \hat{Q})$, which implies our claim by induction.

Assume without loss of generality that $(P_1, Q_1) = (\hat{P}, \hat{Q})$ and $(P_2, Q_2) = (\tilde{P}_2, \tilde{Q}_2) \neq (\hat{P}, \hat{Q})$. We can rewrite $f_1(\mathbf{P}, \mathbf{Q})$ as

$$\frac{1}{N-1}\left(2(k_1 - 1)\operatorname{trace}(\hat{P}\hat{Q}) + \sum_{i \neq 1} k_i \operatorname{trace}(\hat{P}\tilde{Q}_i + \tilde{P}_i\hat{Q})\right), \tag{2}$$

and $f_2(\mathbf{P}, \mathbf{Q})$ as

$$\frac{1}{N-1}\left(2(k_2 - 1)\operatorname{trace}(\tilde{P}_2\tilde{Q}_2) + \sum_{i \neq 1} k_i \operatorname{trace}(\tilde{P}_2\tilde{Q}_i + \tilde{P}_i\tilde{Q}_2)\right). \tag{3}$$

Subtracting (3) from (2) and rearranging gives

$$\begin{aligned} &\frac{k_1 - 1}{N-1}\operatorname{trace}\left((\hat{P} - \tilde{P}_2)\hat{Q} + \hat{P}(\hat{Q} - \tilde{Q}_2)\right) + \frac{k_2 - 1}{N-1}\operatorname{trace}\left((\hat{P} - \tilde{P}_2)\tilde{Q}_2 + \tilde{P}_2(\hat{Q} - \tilde{Q}_2)\right) \\ &\qquad + \frac{1}{N-1}\sum_{i \notin \{1,2\}} \operatorname{trace}\left((\hat{P} - \tilde{P}_2)\tilde{Q}_i + \tilde{P}_i(\hat{Q} - \tilde{Q}_2)\right). \end{aligned} \tag{4}$$

Let $e_P \geq 0$ be the number of rows in $\hat{P}$ that do not match $\tilde{P}_2$ and similarly let $e_Q \geq 0$ be the number of rows in $\hat{Q}$ that do not match $\tilde{Q}_2$. The quantity (4) is greater than or equal to

$$\frac{k_1 - 1}{N-1}(e_P + e_Q) - \frac{k_2 - 1}{N-1}(e_P + e_Q) - \frac{e_P + e_Q}{N-1}\sum_{i \notin \{1,2\}} k_i,$$

or

$$\frac{e_P + e_Q}{N-1}\left((k_1 - 1) - (k_2 - 1) - (N - k_2 - k_1)\right). \tag{5}$$

Simplifying (5) gives

$$\frac{e_P + e_Q}{N-1}(2k_1 - N) \geq \frac{e_P + e_Q}{N-1}(2(\lfloor N/2 \rfloor + 1) - N) > 0,$$

as required, noting that the $(\tilde{P}_2, \tilde{Q}_2)$ was an arbitrary language distinct from $(\hat{P}, \hat{Q})$.

We have thus far shown that so long as a single aligned langauge is used by a strict majority of the population, the unperturbed process always returns to the monomorphic state in that aligned language. It follows that at least $\lceil N/2 \rceil$ mutations are required to reach a new absorbing state, implying $\min_{k \neq s} r_{sk} \geq \lceil N/2 \rceil$.

Consider again the optimal state $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$. Let (P, Q) be any aligned language other than $(\hat{P}, \hat{Q})$. Suppose that $(\mathbf{P}, \mathbf{Q})$ consists of $\lceil N/2 \rceil$ users of (P, Q), with the remaining $\lfloor N/2 \rfloor$ players using $(\hat{P}, \hat{Q})$. Assume without loss of generality that $(P_1, Q_1) = (P, Q)$ and $(P_2, Q_2) = (\hat{P}, \hat{Q})$. We claim that $1 \in \operatorname{argmax}_k f_k(\mathbf{P}, \mathbf{Q})$, so that the unperturbed process can experience an increase in the number of (P, Q) users with positive probability due to (1). To see this, rewrite $f_1(\mathbf{P}, \mathbf{Q})$ as

$$\frac{1}{N-1}\Big(2(\lceil N/2 \rceil - 1)\operatorname{trace}(PQ) + (\lfloor N/2 \rfloor)\operatorname{trace}(P\hat{Q} + \hat{P}Q)\Big), \tag{6}$$

and $f_2(\mathbf{P}, \mathbf{Q})$ as

$$\frac{1}{N-1}\Big(2(\lfloor N/2 \rfloor - 1)\operatorname{trace}(\hat{P}\hat{Q}) + (\lceil N/2 \rceil)\operatorname{trace}(P\hat{Q} + \hat{P}Q)\Big). \tag{7}$$

Subtracting (7) from (6) and rearranging gives

$$\frac{\lceil N/2 \rceil - \lfloor N/2 \rfloor}{N-1}\Big(2m - \operatorname{trace}(P\hat{Q} + \hat{P}Q)\Big) \geq 0,$$

as required. We now have $\lceil N/2 \rceil + 1 \geq \lfloor N/2 \rfloor + 1$ users of aligned language (P, Q), so that the unperturbed process must procede to the optimal state containing only (P, Q) due to our previous argument. Since (P, Q) is an arbitrary aligned language other than $(\hat{P}, \hat{Q})$ the lemma follows. ■

In order to use Lemma 4.3.1 in the proof of Theorem 4.3.2 we also need to know $\min_{k \neq j} r_{jk}$ when $R_j \notin \mathcal{O}$, which we now take up.

Lemma 4.3.3 *For any $R_s \notin \mathcal{O}$ we have $\min_{k \neq s} r_{sk} = 1$ and there exists $R_t \in \mathcal{O} - R_s$ such that $t \in \operatorname{argmin}_{k \neq s} r_{sk}$.*

Proof: The proof is constructive, and also works for $m > n$. Let $R_s = (\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ be the monomorphic state in the language $(\hat{P}, \hat{Q})$, which is not aligned. We will show that there exists an aligned language $(\tilde{P}, \tilde{Q})$ whose associated monomorphic state $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$ can be reached from R_s via a single mutation. First, we provide an example. Suppose

$$(\hat{P}, \hat{Q}) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right),$$

and consider

$$(\tilde{P}, \tilde{Q}) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right).$$

Let $(\mathbf{P}, \mathbf{Q})$ satisfy $(P_k, Q_k) = (\tilde{P}, \tilde{Q})$ for $k \leq K$, with $1 \leq K < N$. Further, let $(P_k, Q_k) = (\hat{P}, \hat{Q})$ for $k > K$. We will show that $\{k \leq K\} = \operatorname{argmax}_k f_k(\mathbf{P}, \mathbf{Q})$ so that, due to (1), the unperturbed process must increase K. It follows from induction that the resistance from $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ to $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$ is equal to one. To see this, rewrite $f_1(\mathbf{P}, \mathbf{Q})$ as

$$\frac{2(K-1)}{N-1} \operatorname{trace}(\tilde{P}\tilde{Q}) + \frac{N-K}{N-1} \operatorname{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q}), \tag{8}$$

and $f_N(\mathbf{P}, \mathbf{Q})$ as

$$\frac{2(N-K-1)}{N-1} \operatorname{trace}(\hat{P}\hat{Q}) + \frac{K}{N-1} \operatorname{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q}). \tag{9}$$

Subtracting (9) from (8) and substituting into the trace$(\cdot)$ terms gives

$$\frac{N+2K-4}{N-1} > 0,$$

as required. We next illustrate a general procedure for producing languages like $(\tilde{P}, \tilde{Q})$.

Let the set

$$\mathcal{K} \equiv \{i : \sum_k \hat{P}_{ik} \hat{Q}_{ki} = 1\}, \tag{10}$$

indicate the rows of $\hat{P}$ (columns of $\hat{Q}$) that contribute to trace$(\hat{P}\hat{Q})$. As a first step, let

$$\tilde{P}^0_{ij} = \begin{cases} \hat{P}_{ij}, & i \in \mathcal{K} \\ \hat{Q}_{ji}, & i \notin \mathcal{K}, \sum_k \hat{Q}_{ki} = 1 \\ 0, & \text{otherwise} \end{cases}, \tag{11}$$

so that

$$\text{trace}(\tilde{P}^0 \hat{Q}) \geq \sum_{i \in \mathcal{K}} \sum_j \hat{P}_{ij} \hat{Q}_{ji} = \text{trace}(\hat{P}\hat{Q}),$$

where the inequality follows from (11) and the equality follows from (10). The matrix $\tilde{P}^0$ is both column sub-stochastic[5] and row sub-stochastic. Therefore, let $\tilde{P}$ be any matrix that is binary, row stochastic, has no zero columns, and satisfies $\tilde{P}_{ij} \geq \tilde{P}^0_{ij}$ for every i, j[6]. Next, let $\tilde{Q}$[7] be any binary, row stochastic, and column sub-stochastic matrix satisfying

$$\tilde{P}^0_{ji} \leq \tilde{Q}_{ij} \leq \tilde{P}_{ji} \quad \forall i, j. \tag{12}$$

The reasoning behind (12) is as follows. We start from $\tilde{P}'$, which is column stochastic, but may have rows summing to more than one, thereby violating row stochasticity. We need to replace each row summing to more than one with a standard basis vector. This can clearly be done in such a manner that the second inequality in (12) is satisfied by restricting ourselves to removing existing ones from $\tilde{P}'$. Since $\tilde{P}_{ij} \geq \tilde{P}^0_{ij}$, the first inequality can always be satisfied as well. Thus, we have

$$\text{trace}(\hat{P}\tilde{Q}) \geq \text{trace}(\hat{P}(\tilde{P}^0)') \geq \sum_{i \in \mathcal{K}} \sum_j \hat{P}_{ij} \hat{P}_{ij} = \text{trace}(\hat{P}\hat{Q}).$$

[5] In other words, column sums of $\tilde{P}^0$ are less than or equal to one.
[6] When $m = n$, $\tilde{P}$ is a permutation matrix.
[7] When $m = n$, we can simply let $\tilde{Q} = \tilde{P}'$.

We also have $\text{trace}(\tilde{P}\tilde{Q}) = n$. To see this, assume the contrary so that $\text{trace}(\tilde{P}\tilde{Q}) < n$. Then, by column sub-stochasticity of $\tilde{Q}$, there exists i, j such that $Q_{ij} = 1$ while $P_{ji} = 0$, contradicting (12).

As in our example, let $(\mathbf{P}, \mathbf{Q})$ satisfy $(P_k, Q_k) = (\tilde{P}, \tilde{Q})$ for $k \leq K$, with $1 \leq K < N$ and let $(P_k, Q_k) = (\hat{P}, \hat{Q})$ for $k > K$. Using (8) and (9) we can rewrite $f_1(\mathbf{P}, \mathbf{Q}) - f_N(\mathbf{P}, \mathbf{Q})$ as

$$\frac{K}{N-1}\Big(2\,\text{trace}(\tilde{P}\tilde{Q}) - \text{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q})\Big) + \frac{N-K-1}{N-1}\Big(\text{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q}) - 2\,\text{trace}(\hat{P}\hat{Q})\Big)$$
$$+ \frac{1}{N-1}\Big(\text{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q}) - 2\,\text{trace}(\tilde{P}\tilde{Q})\Big). \tag{13}$$

The second term of (13) is non-negative, so, combining the first and third terms, we have that (13) is greater than or equal to

$$\Big(2\,\text{trace}(\tilde{P}\tilde{Q}) - \text{trace}(\tilde{P}\hat{Q} + \hat{P}\tilde{Q})\Big)\frac{K-1}{N-1} \geq 0,$$

so that $1 \in \text{argmax}_k\, f_k(\mathbf{P}, \mathbf{Q})$ for any $K \geq 1$. It follows that a single mutation is sufficient to reach $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$ from $R_s = (\hat{\mathbf{P}}, \hat{\mathbf{Q}})$, which concludes the proof of the lemma. ■

Combining Lemma 4.3.1, Lemma 4.3.2 and Lemma 4.3.3, we have for each $R_i \in \mathcal{O}$

$$\gamma(R_i) \geq \lambda(R_i) = \lceil N/2 \rceil(|\mathcal{O}| - 1) + (J - |\mathcal{O}|), \tag{14}$$

while for $R_i \notin \mathcal{O}$ we have

$$\gamma(R_i) \geq \lambda(R_i) = \lceil N/2 \rceil(|\mathcal{O}|) + (J - |\mathcal{O}| - 1).$$

The next lemma establishes that (14) is tight so that if $R_i \in \mathcal{O}$ and $R_j \notin \mathcal{O}$ we have

$$\gamma(R_i) = \lambda(R_i) \equiv \gamma_{\mathcal{O}} < \lambda(R_j) \leq \gamma(R_j),$$

where $\gamma_{\mathcal{O}}$ is the stochastic potential of each state in $\mathcal{O}$. Thus, by Theorem 2.2.1, the stochastically stable states of $\mathcal{P}^{\epsilon}_{m,m}$ are precisely the states in $\mathcal{O}$, which proves Theorem 4.3.2. ■

Lemma 4.3.4 *For any $R_j \in \mathcal{O}$ we have*

$$\gamma(R_i) = \lceil N/2 \rceil(|\mathcal{O}| - 1) + (J - |\mathcal{O}|) \equiv \gamma_{\mathcal{O}}$$

Proof: By (14) we have $\gamma(R_i) \geq \gamma_O$, so what remains is to show that an R_j-tree achieving γ_O always exists. We describe such a tree with a depth of two. For $R_i \notin \mathcal{O}$, Lemma 4.3.3 indicates that there is a state in $\mathcal{O}$ that is reached with resistance equal to one. Thus, one level of the tree consists of links from states not in $\mathcal{O}$ to states in $\mathcal{O}$. The next level is simply links from each $R_k \in \mathcal{O}, k \neq i$ to R_j. By Lemma 4.3.2, these links all have resistance equal to one as well. ■

The next section develops a result analogous to Theorem 4.3.2 for the case of $m \neq n$.

4.3.4 Efficiency of $\mathcal{P}^{\epsilon}_{m,n}$ when $m \neq n$

From the perspective of stochastic stability, the long-run behavior of $\mathcal{P}^{\epsilon}_{m,n}$ is the same whether $m \neq n$ or $m = n$.

Theorem 4.3.3 *A state of $\mathcal{P}^{\epsilon}_{m,n}$, where $m \neq n$, is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states .*

The remainder of this section develops the proof of Theorem 4.3.3. We assume without loss of generality that $m > n$ whenever $m \neq n$. The argument is similar to that given for Theorem 4.3.2, with Lemma 4.3.1 again being the workhorse. Thus, we need to again find $min_{k \neq j} r_{jk}$ for different absorbing states R_j. For $R_j \notin \mathcal{O}$ we inherit Lemma 4.3.3, which nowhere assumed that $m = n$. Thus, we have $min_{k \neq j} r_{jk} = 1$ when $R_j \in \mathcal{O}$. The next lemma shows that the same is true of $R_j \in \mathcal{O}$.

Lemma 4.3.5 *For any $R_s \in \mathcal{O}$ we have $min_{k \neq s} r_{sk} = 1$.*

Proof: Let $R_s = (\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ be the monomorphic state in the language $(\hat{P}, \hat{Q})$. Since $n < m$ and $\hat{Q}$ is row stochastic, $\hat{Q}$ must have a zero column. Let k_1 be the index of this zero column and let k_2 satisfy $\hat{P}_{k_1 k_2} \neq 1$. We define

$$\tilde{P}_{ij} = \begin{cases} \hat{P}_{ij}, & i \neq k_1 \\ 1, & i = k_1, j = k_2 \\ 0, & i = k_1, j \neq k_2 \end{cases},$$

so that trace$(\tilde{P}\hat{Q}) = n$. Any state that includes users of both $(\hat{P}, \hat{Q})$ and $(\tilde{P}, \hat{Q})$, but no other languages assigns the payoff n to each and every player. Thus, the monomorphic state in either language can be reached without further resistance. The lemma is thus proven by our now familiar inductive argument. ■

Combining Lemma 4.3.1, Lemma 4.3.3, and Lemma 4.3.5 we have that for any R_j the stochastic potential

$$\gamma(R_j) \geq \lambda(R_j) = J - 1. \tag{15}$$

For $R_j \notin \mathcal{O}$, the inequality (15) is strict.

Lemma 4.3.6 *For any $R_j \notin \mathcal{O}$ we have $\gamma(R_j) > J - 1$.*

Proof: Let $R_j = (\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ be the monomorphic state in the language $(\hat{P}, \hat{Q})$. In order to achieve $\gamma(R_j) = J + |\mathcal{O}| - 1$, there must be an R_j-tree that achieves a resistance of one at every edge. To see that this is not possible observe that every R_j-tree must have at least one edge that emanates from a state in $\mathcal{O}$ and terminates at a state not in $\mathcal{O}$. This edge has resistance strictly greater than one if there exists no language $(\tilde{P}, \tilde{Q})$ such that when $(P_1, Q_1) = (\tilde{P}, \tilde{Q})$ and $(P_i, Q_i) = (\hat{P}, \hat{Q}), i \neq 1$ we have $f_1(\mathbf{P}, \mathbf{Q}) \geq f_N(\mathbf{P}, \mathbf{Q})$. From (13) with $K = 1$, noting that in this case the second term will be negative, we have that $f_1(\mathbf{P}, \mathbf{Q}) - f_N(\mathbf{P}, \mathbf{Q}) < 0$. ■

The proof is complete once we show that for $R_j \in \mathcal{O}$, the inequality (15) is satisfied with equality. This requires an R_j-tree with a resistance of one for every edge. The construction of such trees utilizes the following lemma.

Lemma 4.3.7 *For any two distinct states $R_s, R_t \in \mathcal{O}$ there exists $K \geq 1$, and a sequence of states $\{R_{i_k}\}_{k=1}^{K}$ such that*

1. *$i_1 = s$,*

2. *$i_K = t$,*

3. *$R_{i_k} \in \mathcal{O}$ for each $k \in \{1, ..., K\}$,*

4. $r_{i_k i_{k+1}} = 1$ *for each* $k \in \{1, ..., K-1\}$.

The proof of Lemma 4.3.7 is somewhat lengthy and can be found in Section 4.8 at the end of the chapter. The proof of Theorem 4.3.3 is concluded with the next lemma.

Lemma 4.3.8 *For any* $R_j \in \mathcal{O}$ *we have* $\gamma(R_j) = \lambda(R_j) = J - 1$.

Proof: We need to show that for any $R_j \in \mathcal{O}$ we can construct an R_j-tree with each edge having resistance equal to one. For $R_i \notin \mathcal{O}$ we can, by Lemma 4.3.3, reach some state in $\mathcal{O}$ with resistance equal to one. From $R_i \in \mathcal{O}, R_i \neq R_j$, we can, by Lemma 4.3.7, construct a path to R_j via states in $\mathcal{O}$ with each edge having resistance equal to one. Any redundancies can be eliminated as we go because all R_j-trees have precisely $J - 1$ edges. ■

4.3.5 Discussion

Theorem 4.3.2 and Theorem 4.3.3 provide that the dynamic process we have described, encompassing both mutation and selection, manages to coordinate exclusively on states that maximize linguistic coherence. The particular language will nevertheless change from time to time, consistent with observed phenomenon of linguistic drift [62]. In the more natural case of $m \neq n$, where the number of objects and symbols does not match, this drift ought to be particularly prevalent as the necessary ambiguity provides pathways for such changes, as described in Lemma 4.3.7. The many well documented cognates in modern natural languages point to the divergence in the meaning of a particular form as a vehicle for linguistic change [65].

In short, the intrinsic randomness that enables the players to search the set of languages will prevent settling into any sort of permanent language state. However, despite never freezing in a particular language, we can expect players to understand each other as well as possible for a high proportion of the time.

The next section introduces several variations on the dynamics.

4.4 Variations on the dynamics

Up until this point we have studied the dynamics described by (1), which are somewhat contrived in that, absent mutations, only the most fit players are able to reproduce themselves. In this section we will show that stochastic stability of optimal states is preserved under a number of more realistic variations on the dynamics.

4.4.1 Twice-perturbed dynamics

One mechanism to enable players other than the most fit to reproduce some of the time is modification of the perturbation in (1). At each time t we select a player i according to F, as before. The selected player chooses a candidate language

$$(\hat{P}, \hat{Q}) = \begin{cases} (P_{\hat{k}}, Q_{\hat{k}}), & \text{with probability } 1 - \rho\epsilon \\ (P_{\tilde{k}}, Q_{\tilde{k}}), & \text{with probability } \rho\epsilon \end{cases}, \tag{16}$$

where $\rho \in (0, 1/\epsilon)$, $\tilde{k} = \text{rand}(\{1, 2, ..., N\})$ and once again $\hat{k} \in \text{argmax}_k\, f_k(\mathbf{P}[t-1], \mathbf{Q}[t-1])$. She then updates her language as

$$(P_i[t], Q_i[t]) = \begin{cases} (\hat{P}, \hat{Q}), & \text{with probability } 1 - \epsilon \\ \text{rand}(\mathcal{L}_{m,n}), & \text{with probability } \epsilon \end{cases},$$

while the other players stand pat with

$$(P_j[t], Q_j[t]) = (P_j[t-1], Q_j[t-1]) \quad \forall j \neq i.$$

Let $\mathcal{P}^{\epsilon}_{m,n,\mathtt{tp}}$ refer to the regular perturbed Markov process induced by these dynamics. The analysis of $\mathcal{P}^{\epsilon}_{m,n,\mathtt{tp}}$ follows directly from that of $\mathcal{P}^{\epsilon}_{m,n}$, so we state the main result as a corollary to Theorem 4.3.2 and Theorem 4.3.3.

Corollary 4.4.1 *A state of $\mathcal{P}^{\epsilon}_{m,n,\mathtt{tp}}$ is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states .*

Idea of the proof: While $\mathcal{P}^{\epsilon}_{m,n,\mathtt{tp}}$ and $\mathcal{P}^{\epsilon}_{m,n}$ are clearly different, we have $\mathcal{P}^{0}_{m,n,\mathtt{tp}} = \mathcal{P}^{\epsilon}_{m,n}$. Thus they have the same set of recurrent communication classes and the same resistances between those classes. In fact, $\mathcal{P}^{\epsilon}_{m,n}$ does afford reproductive opportunities to players not among the most fit, but only through rare mutations. This is still true in $\mathcal{P}^{\epsilon}_{m,n,\mathrm{tp}}$, but with such events being more (or less) probable than other mutations by an arbitrary constant factor. Scaling of perturbation terms by positive, constant factors does not influence resistances, which can be seen by inspecting (1). ■

Using multiple, more elaborate scaling factors in the manner of (16) can bring $\mathcal{P}^{\epsilon}_{m,n}$ even closer to the replicator dynamics, which provides reproductive opportunities in proportion to fitness. We do not develop such a dynamic explicitly here, although stochastic stability of $\mathcal{O}$ would follow from the same argument offered above. We note however, that conclusions would still only be valid in the small ϵ limit. Thus, the absolute probability that a player not among the most fit reproduces is still potentially quite small.

In order to arrive at significantly more realistic dynamics that preserve the stochastic stability properties, substantial structural changes are required. The next sections describes such an approach.

4.4.2 Pairwise competition dynamics

Inspired by pairwise comparison dynamics [70], we propose an intuitive new dynamic. In pairwise competition dynamics we select two players i and j with $i \neq j$ at each time t according to a probability distribution over pairs that is parametrized by $(\mathbf{P}[t-1], \mathbf{Q}[t-1])$. That is, (i, j) is distributed as $F(\mathbf{P}[t-1], \mathbf{Q}[t-1])$, where F is any family of full support distributions, noting slight abuse of notation. Assume without loss of generality that $f_i(\mathbf{P}[t-1], \mathbf{Q}[t-1]) \leq f_j(\mathbf{P}[t-1], \mathbf{Q}[t-1])$ and let

$$(P_i[t], Q_i[t]) = \begin{cases} (P_j[t], Q_j[t], & \text{with probability } 1-\epsilon \\ \mathrm{rand}(\mathcal{L}_{m,n}), & \text{with probability } \epsilon \end{cases}, \tag{17}$$

with all players other than i standing pat. We interpret (17) as localized competition over reproductive opportunities and resources such as food, nesting sites, and mates. Absent mutation the fitter player reproduces herself, while the less fit player dies out. The fitness function itself still reflects a global interaction— effective communication with the broader population confers advantages in local competitions. Let $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pc}}$ refer to the regular perturbed Markov process induced by pairwise competition dynamics. We have the following result, stated without proof, and also a corollary of Theorem 4.3.2 and Theorem 4.3.3.

Corollary 4.4.2 *A state of $\mathcal{P}^{\epsilon}_{m,n,pc}$ is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states.*

The proof of Corollary 4.4.2 is essentially the same as that of Theorem 4.3.2 and 4.3.3. While in this case even the unperturbed process is different, the recurrent communication classes and minimum resistance paths remain the same.

A variation on pairwise competition dynamics is pairwise competition dynamics on a fixed graph. That is, we fix a connected graph G whose vertex set is the set of players $\{1, 2, ..., N\}$ and select only pairs of players from the edge set of G. We then proceed as in pairwise competition dynamics. Let $\mathcal{P}^{\epsilon}_{m,n,G}$ refer to the regular perturbed Markov process induced by pairwise competition dynamics on the fixed graph G. Not surprisingly $\mathcal{P}^{\epsilon}_{m,n,G}$ follows our now familiar theme.

Corollary 4.4.3 *A state of $\mathcal{P}^{\epsilon}_{m,n,G}$ is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states.*

The proof of Corollary 4.4.3 is again the same as before, noting that any state that is not monormorphic must have some edge between two players speaking different languages.

The variations on the dynamics we have thus far considered have all utilized the same model of mutation. At each time t, the player granted a revision opportunity adopts a random language with probability ϵ. In the next section we consider instead point mutations, which assume that mutations only randomize within a neighbourhood of the nominal language.

4.5 Point mutations

In general, we expect that mistakes or mutations in the updating process ought to lead to only small, localized changes to the languages that would have otherwise been chosen. In this section we consider a variation on our dynamic model that restricts mutations to jumps that are "close" to the nominal parent language in a precise sense. In particular, we suppose that at each discrete time instant t, a player i is randomly selected for a strategy revision opportunity according to $F(\mathbf{P}[t-1], \mathbf{Q}[t-1]) > 0$, as before. The selected player updates her strategy as

$$(P_i[t], Q_i[t]) = \begin{cases} (P_{\hat{k}}, Q_{\hat{k}}), & \text{with probability } 1-\epsilon \\ \text{rand}\left(B_1^H(P_{\hat{k}}, Q_{\hat{k}})\right), & \text{with probability } \epsilon \end{cases},$$

where $B_1^H(P_{\hat{k}}, Q_{\hat{k}})$ is the ball of Hamming distance one centered at $(P_{\hat{k}}, Q_{\hat{k}})$, given by

$$\{(P, Q) \in \mathcal{L}_{m,n} : \frac{1}{2}\sum_i \sum_j \left(|P_{ij} - (P_{\hat{k}})_{ij}| + |Q_{ji} - (Q_{\hat{k}})_{ji}|\right) \leq 1\}.$$

Put differently, mutations modify at most one row of one of the matrices of the language that otherwise would have been used. As usual, the other players stand pat. We define the element-wise Hamming distance between two languages as

$$d_H\left((P, Q), (\hat{P}, \hat{Q})\right) = \frac{1}{2}\sum_i \sum_j \left(|\hat{P}_{ij} - P_{ij}| + |\hat{Q}_{ji} - Q_{ji}|\right)$$

Let $\mathcal{P}^{\epsilon}_{m,n,\text{pm}}$ refer to the regular perturbed Markov process just described. We find that the behavior of this process is qualitatively consistent with those considered thus far.

Theorem 4.5.1 *A state of $\mathcal{P}^{\epsilon}_{m,n,pm}$ is stochastically stable if and only if it is contained in $\mathcal{O}$, the set of optimal states.*

The rest of this section provides the proof of Theorem 4.5.1. Since $\mathcal{P}^0_{m,n,\text{pm}} = \mathcal{P}^0_{m,n}$, the recurrent communication classes are still the monomorphic states. We will want to once again make use of Lemma 4.3.1. First consider $m > n$. Note that the sequences constructed in Lemma 4.3.7 always involved modifications to one row at a time, so the lemma holds for $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pm}}$ too. Thus for any state $R_s \in \mathcal{O}$ we can construct a path in $\mathcal{O}$ to any other state in $\mathcal{O}$ that achieves unity resistance at every edge. From states not in $\mathcal{O}$ we would like to piggyback on Lemma 4.3.3, but we must first make a modification, which we present as a new lemma.

Lemma 4.5.1 *Suppose $m > n$, then for any $R_s \notin \mathcal{O}$ there exists $R_t \in \mathcal{O}$, $K \geq 2$, and a sequence of states $\{R_{i_k}\}_{k=1}^{K}$ such that*

1. $i_1 = s$,

2. $i_K = t$,

3. $r_{i_k i_{k+1}} = 1$ *for each* $k \in \{1, ..., K-1\}$.

Proof: Recall that given a monomorphic state $(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) \notin \mathcal{O}$, Lemma 4.3.3 constructs a monomorphic state $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}) \in \mathcal{O}$ that can be reached with resistance equal to one. Let $(\tilde{P}, \tilde{Q})$ and $(\hat{P}, \hat{Q})$ be, respectively, the languages that $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$ and $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ are monomorphic in. It is possible that $(\tilde{P}, \tilde{Q}) \notin B_1^H(\hat{P}, \hat{Q})$, so the resistance between these states is potentially greater than one for $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pm}}$. To resolve this issue we break up the transition from $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ to $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$ into a sequence with the resistances between intermediate monomorphic states all equal to one. This can be achieved by constructing a sequence of languages

$$(\tilde{P}^0, \tilde{Q}^0) \equiv (\hat{P}, \hat{Q}), (\tilde{P}^1, \tilde{Q}^1), (\tilde{P}^2, \tilde{Q}^2), ..., (\tilde{P}^L, \tilde{Q}^L) \equiv (\tilde{P}, \tilde{Q})$$

where $L = d_H\left((\tilde{P}, \tilde{Q}), (\hat{P}, \hat{Q})\right)$ and for each $1 \leq k \leq L$ we have

$$d_H\left((\tilde{P}^k, \tilde{Q}^k), (\hat{P}, \hat{Q})\right) = k \tag{18}$$

and

$$d_H\left((\tilde{P}^k, \tilde{Q}^k), (\tilde{P}, \tilde{Q})\right) = L - k. \tag{19}$$

We argue inductively that the monomorphic states associated with any such sequence suffice. To this end, consider a partial sequence

$$(\tilde{P}^0, \tilde{Q}^0) \equiv (\hat{P}, \hat{Q}), (\tilde{P}^1, \tilde{Q}^1), (\tilde{P}^2, \tilde{Q}^2), ..., (\tilde{P}^l, \tilde{Q}^l),$$

where $0 \leq l < L$. We assume this sequence satisfies (18) and (19) and that the resistance between the monomorphic states associated with any pair from left to right is unity. The base case of $l = 0$ is trivial. For the induction step assume we have such a sequence up to $l < L$, and extend the sequence by adding any language $(\tilde{P}^{l+1}, \tilde{Q}^{l+1})$ satisfying (18) and (19). Let $R_{\hat{s}} = (\tilde{\mathbf{P}}^l, \tilde{\mathbf{Q}}^l)$ and $R_{\hat{t}} = (\tilde{\mathbf{P}}^{l+1}, \tilde{\mathbf{Q}}^{l+1})$, the monomorphic states in, respectively,$(\tilde{P}^l, \tilde{Q}^l)$ and $(\tilde{P}^{l+1}, \tilde{Q}^{l+1})$. We need to show that $r_{\hat{s}\hat{t}} = 1$, which follows from the arguments given at the end of Lemma 3 once we show that $\text{trace}(\tilde{P}^{l+1}\tilde{Q}^l) \geq \text{trace}(\tilde{P}^l\tilde{Q}^l)$ and $\text{trace}(\tilde{P}^l\tilde{Q}^{l+1}) \geq \text{trace}(\tilde{P}^l\tilde{Q}^l)$.

Either $\tilde{P}^{l+1} = \tilde{P}^l$ or $\tilde{Q}^{l+1} = \tilde{Q}^l$. We first consider $\tilde{P}^{l+1} = \tilde{P}^l$. In this case we only need to show $\text{trace}(\tilde{P}^l\tilde{Q}^{l+1}) \geq \text{trace}(\tilde{P}^l\tilde{Q}^l)$. The two matrices $\tilde{Q}^l$ and $\tilde{Q}^{l+1}$ differ in a single row, which we will index with $\tilde{i}$. Let $\tilde{j}$ be the unique index satisfying $\tilde{Q}^{l+1}_{\tilde{i}\tilde{j}} = 1$. First, suppose $\tilde{P}^l_{\tilde{j}\tilde{i}} = \tilde{P}_{\tilde{j}\tilde{i}}$, then

$$\text{trace}\left(\tilde{P}^l(\tilde{Q}^{l+1} - \tilde{Q}^l)\right) = \sum_j \tilde{P}_{j\tilde{i}}(\tilde{Q}_{\tilde{i}j} - \hat{Q}_{\tilde{i}j}) = 1 - \sum_j \tilde{P}_{j\tilde{i}}\hat{Q}_{\tilde{i}j} \geq 0,$$

as required. If instead $\tilde{P}^l_{\tilde{j}\tilde{i}} = \hat{P}_{\tilde{j}\tilde{i}}$, then we have

$$\text{trace}\left(\tilde{P}^l(\tilde{Q}^{l+1} - \tilde{Q}^l)\right) = \sum_j \hat{P}_{j\tilde{i}}(\tilde{Q}_{\tilde{i}j} - \hat{Q}_{\tilde{i}j}) \geq \hat{P}_{\tilde{j}\tilde{i}}\tilde{Q}_{\tilde{i}\tilde{j}} - \hat{P}_{\tilde{j}\tilde{i}}\hat{Q}_{\tilde{i}\tilde{j}},$$

where $\hat{j}$ is the unique index satsifying $\hat{Q}_{\tilde{i}\hat{j}} = 1$. If $\hat{P}_{\tilde{j}\tilde{i}}\hat{Q}_{\tilde{i}\tilde{j}} = 1$ then by (10),(11), and (12) we have $\tilde{Q}_{\tilde{i}\hat{j}} \geq \hat{P}_{\tilde{j}\tilde{i}} = \hat{Q}_{\tilde{i}\hat{j}}$, as required.

Next, suppose $\tilde{Q}^{l+1} = \tilde{Q}^l$. In this case we only to need to show $\text{trace}(\tilde{P}^{l+1}\tilde{Q}^l) \geq \text{trace}(\tilde{P}^l\tilde{Q}^l)$. The two matrices $\tilde{P}^l$ and $\tilde{P}^{l+1}$ differ in a single row, which we will index with $\tilde{i}$. Let $\tilde{j}$ be the unique index satisfying $\tilde{P}^{l+1}_{\tilde{i}\tilde{j}} = 1$. First, suppose $\tilde{Q}^l_{\tilde{j}\tilde{i}} = \tilde{Q}_{\tilde{j}\tilde{i}}$, then

$$\text{trace}\left((\tilde{P}^{l+1} - \tilde{P}^l)\tilde{Q}^l\right) = \sum_j (\tilde{P}_{\tilde{i}j} - \hat{P}_{\tilde{i}j})\tilde{Q}_{j\tilde{i}} = \sum_j \tilde{Q}_{j\tilde{i}} - \sum_j \hat{P}_{\tilde{i}j}\tilde{Q}_{j\tilde{i}} \geq 0,$$

as required. If instead $\tilde{Q}^l_{\tilde{j}\tilde{i}} = \hat{Q}_{\tilde{j}\tilde{i}}$, then we have

$$\operatorname{trace}\left((\tilde{P}^{l+1} - \tilde{P}^l)\tilde{Q}^l\right) = \sum_j (\tilde{P}_{\tilde{i}j} - \hat{P}_{\tilde{i}j})\hat{Q}_{j\tilde{i}} \geq \tilde{P}_{\tilde{i}\hat{j}}\hat{Q}_{\tilde{j}\tilde{i}} - \hat{P}_{\tilde{i}\hat{j}}\hat{Q}_{\tilde{j}\tilde{i}},$$

where $\hat{j}$ is the unique index satisfying $\hat{Q}_{\tilde{j}\tilde{i}} = 1$. If $\hat{P}_{\tilde{i}\hat{j}}\hat{Q}_{\tilde{j}\tilde{i}} = 1$ then by (10) and (11) we have $\tilde{P}_{\tilde{i}\hat{j}} \geq \hat{P}_{\tilde{i}\hat{j}}$, as required. ■.

Thus far we have established that for $R_j \in \mathcal{O}$ we have

$$\gamma(R_j) \geq \lambda(R_j) = J - 1,$$

where the inequality follows from Lemma 4.3.1. The bound is tight.

Lemma 4.5.2 *Suppose $m > n$, then for any $R_j \in \mathcal{O}$ we have $\gamma(R_j) = \lambda(R_j) = J - 1$.*

Proof: We need to show that for any $R_j \in \mathcal{O}$ we can construct an R_j-tree with each edge having resistance equal to one. For $R_i \notin \mathcal{O}$ we can, by Lemma 4.5.1, reach some state in $\mathcal{O}$ with each resistance along the path equal to one. Redundancies can be eliminated as we go. From $R_i \in \mathcal{O}, R_i \neq R_j$, we can, by Lemma 4.3.7, construct a path to R_j via states in $\mathcal{O}$ with each edge having resistance equal to one. Again, any redundancies can be eliminated as we go because all R_j-trees have precisely $J - 1$ edges. ■

Resistances between absorbing states in $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pm}}$ are always greater than or equal to the corresponding resistances in $\mathcal{P}^{\epsilon}_{m,n}$ because the latter does not restrict the allowed mutations. Lower bounds on resistances derived for $\mathcal{P}^{\epsilon}_{m,n}$, in particular Lemma 4.3.5, continue to be valid for $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pm}}$. Thus, for $R_j \notin \mathcal{O}$ we have $\gamma(R_j) > J - 1$, so that the stochastically stable states are precisely $\mathcal{O}$.

Next, consider $m = n$, turning our attention first to the optimal states. Lemma 4.3.2 is not valid for $\mathcal{P}^{\epsilon}_{m,n,\mathrm{pm}}$, neccesitating the following lemma.

Lemma 4.5.3 *Suppose $m = n$, then for any $(\hat{\boldsymbol{P}}, \hat{\boldsymbol{Q}}) \in \mathcal{O}$, any $(\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}) = R_{\bar{\iota}} \in \mathcal{O}$, and any $(\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{Q}}) = R_{\tilde{\iota}} \notin \mathcal{O}$ with $(\tilde{P}, \tilde{Q}) \in B^H_1(\hat{P}, \hat{Q})$ satsifying*

$$d_H\left((\tilde{P}, \tilde{Q}), (\bar{P}, \bar{Q})\right) = d_H\left((\hat{P}, \hat{Q}), (\bar{P}, \bar{Q})\right) - 1,$$

where $(\hat{P},\hat{Q})$, $(\bar{P},\bar{Q})$, *and* $(\tilde{P},\tilde{Q})$ *are, respectively, the languages in* $(\hat{\mathbf{P}},\hat{\mathbf{Q}})$, $(\bar{\mathbf{P}},\bar{\mathbf{Q}})$, *and* $(\tilde{\mathbf{P}},\tilde{\mathbf{Q}})$, *we have* $r_{s\tilde{t}} = \min_{t\neq s} r_{st} = N-1$ *when* N *is even and* $r_{s\tilde{t}} = \min_{t\neq s} r_{st} = N$ *when* N *is odd.*

Proof: First, consider N even. We have

$$\mathrm{trace}(\hat{P}\tilde{Q}+\tilde{P}\hat{Q}) = 2m-1 \geq \mathrm{trace}(P\tilde{Q}+\tilde{P}Q) \tag{20}$$

for any $(P,Q)\in\mathcal{L}_{m,m}$. Next, let $(\bar{P}^0,\bar{Q}^0)\in B_2^H(\hat{P},\hat{Q})$ satisfy $d_H\left((\bar{P}^0,\bar{Q}^0),(\hat{P},\hat{Q})\right)=2$ and

$$\mathrm{trace}(\hat{P}\bar{Q}^0+\bar{P}^0\hat{Q}) = 2n-2 < \mathrm{trace}(\tilde{P}\bar{Q}^0+\bar{P}^0\tilde{Q}) = 2n-1.$$

Consider the state $(\mathbf{P},\mathbf{Q})$, where

$$(P_i,Q_i) = \begin{cases} (\hat{P},\hat{Q}), & i\leq N/2 \\ (\bar{P}^0,\bar{Q}^0), & i\in[N/2+1,N-1] \\ (\tilde{P},\tilde{Q}), & i=N \end{cases}.$$

It is straightforward to see that $\mathcal{P}^{\epsilon}_{m,n,\mathtt{pm}}$ can transition from $(\mathbf{P},\mathbf{Q})$ to $(\tilde{\mathbf{P}},\tilde{\mathbf{Q}}) = R_t$ without resistance so that $r_{st}=N-1$. What remains is to show that this resistance is the minimum achievable. Suppose there exists R_t such that $r_{st}\leq N-2$. This requires that there exist $(\mathbf{P},\mathbf{Q})$ with $(P_1,Q_1)=(P_2,Q_2)=(\hat{P},\hat{Q})$ and

$$\sum_{i=2}^{N} d_H\left((P_i,Q_i),(\hat{P},\hat{Q})\right)\leq N-2,$$

such that there is some

$$\hat{k}\in\operatorname*{argmax}_k f_k(\mathbf{P},\mathbf{Q}),\ \text{with } (P_{\hat{k}},Q_{\hat{k}})\neq(\hat{P},\hat{Q}). \tag{21}$$

In this case there must be two players i_1 and i_2 such that

$$d_H\left((P_{i_1},Q_{i_1}),(\hat{P},\hat{Q})\right) = d_H\left((P_{i_2},Q_{i_2}),(\hat{P},\hat{Q})\right)=1.$$

Otherwise at most $N/2-1$ players have mutated, which precludes (21) by Lemma 4.3.2. We claim that $i_1,i_2\notin\operatorname{argmax}_k f_k(\mathbf{P},\mathbf{Q})$. To see this first note that

$$\mathrm{trace}(P_1Q_j+P_jQ_1)\geq\mathrm{trace}(P_iQ_j+P_jQ_i)-1$$

for all j, and

$$\text{trace}(P_1 Q_j + P_j Q_1) = \text{trace}(P_i Q_j + P_j Q_i) - 1$$

requires $d_H\left((P_j, Q_j), (\hat{P}, \hat{Q})\right) \geq 2$. Taking into account what we know about $(\mathbf{P}, \mathbf{Q})$ we have

$$f_1(\mathbf{P}, \mathbf{Q}) = \frac{1}{N-1}\left(2m + 2(2m-1) + \sum_{j \notin \{1,2,i_1,i_2\}} \text{trace}(P_1 Q_j + P_j Q_1)\right), \tag{22}$$

while

$$f_{i_1}(\mathbf{P}, \mathbf{Q}) \leq \frac{1}{N-1}\left(3(2m-1) + \sum_{j \notin \{1,2,i_1,i_2\}} \text{trace}(P_{i_1} Q_j + P_j Q_{i_1})\right). \tag{23}$$

Subtracting (23) from (22) gives

$$\frac{1}{N-1}\left(1 + \sum_{j \notin \{1,2,i_1,i_2\}} \text{trace}\left((P_1 - P_{i_1})Q_j + P_j(Q_1 - Q_{i_1})\right)\right),$$

with each term in the summation in {-1,0,1}. In order for a term in the summation to be negative we must have $d_H\left((P_j, Q_j), (\hat{P}, \hat{Q})\right) \geq 2$. However, we also must have

$$\sum_{j \notin \{1,2,i_1,i_2\}} d_H\left((P_j, Q_j), (\hat{P}, \hat{Q})\right) \leq N - 4.$$

It follows that for every negative term there must be a positive term, so that the overall summation is non-negative, proving our claim.

Now, $\hat{k}$ in (21) must satisfy $d_H\left((P_{\hat{k}}, Q_{\hat{k}}), (\hat{P}, \hat{Q})\right) \geq 2$. Consider the $N-2$ player state arrived at by removing players i_1 and i_2, denoted $(\mathbf{P}^{N-2}, \mathbf{Q}^{N-2})$. Assume without loss of generality that $(P_1^{N-2}, Q_1^{N-2}) = (\hat{P}, \hat{Q})$ and $(P_{\hat{k}}^{N-2}, Q_{\hat{k}}^{N-2}) = (P_{\hat{k}}, Q_{\hat{k}})$. Due to (20), $f_{\hat{k}}(\mathbf{P}, \mathbf{Q}) \geq f_1(\mathbf{P}, \mathbf{Q})$ implies that in the $N-2$ player game $f_{\hat{k}}(\mathbf{P}^{N-2}, \mathbf{Q}^{N-2}) \geq f_1(\mathbf{P}^{N-2}, \mathbf{Q}^{N-2})$ as well. Proceding like this we eventually obtain a game that contradicts our claim, completing the proof for N even.

Next, consider odd N. Clearly $r_{s\tilde{s}} \leq N$ since the mutations can simply be applied to every player. What remains is to show that N is the minimum achievable resistance. The proof of this fact is essentially the same as for the previous case of N even and is therefore omitted. ■

Lemma 4.5.4 *For any* $(\hat{\boldsymbol{P}}, \hat{\boldsymbol{Q}}) = R_s \notin \mathcal{O}$ *and* $(\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}) = R_t \in \mathcal{O}$ *there exists* $(\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{Q}}) = R_{\tilde{t}}$ *such that* $r_{s\tilde{t}} = 1$ *and*

$$d_H\left((\tilde{P}, \tilde{Q}), (\bar{P}, \bar{Q})\right) = d_H\left((\hat{P}, \hat{Q}), (\bar{P}, \bar{Q})\right) - 1,$$

where $(\hat{P}, \hat{Q})$, $(\bar{P}, \bar{Q})$, *and* $(\tilde{P}, \tilde{Q})$ *are, respectively, the languages in* $(\hat{\boldsymbol{P}}, \hat{\boldsymbol{Q}})$, $(\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}})$, *and* $(\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{Q}})$

Proof: Let the set

$$\mathcal{K}_{\hat{P}} = \{i : \sum_j \hat{P}_{ij}\hat{Q}_{ji} = 1\},$$

and similarly let

$$\mathcal{K}_{\hat{Q}} = \{i : \sum_j \hat{Q}_{ij}\hat{P}_{ji} = 1\},$$

these are the rows making positive contributions to payoffs in, respectively, $\hat{P}$ and $\hat{Q}$. If there is a pair of indices $\hat{i}, \hat{j}$ with $\hat{i} \notin \mathcal{K}_{\hat{P}}$ and $\hat{P}_{\hat{i}\hat{j}} \neq \bar{P}_{\hat{i}\hat{j}}$ then let

$$\tilde{P}_{ij} = \begin{cases} \hat{P}_{ij}, & i \neq \hat{i} \\ \bar{P}_{ij}, & i = \hat{i} \end{cases},$$

and let $\tilde{Q} = \hat{Q}$. In this case, the lemma follows from the same arguments given at the end of Lemma 4.3.3. Alternatively, if there is a pair of indices $\hat{i}, \hat{j}$ with $\hat{i} \notin \mathcal{K}_{\hat{Q}}$ and $\hat{Q}_{\hat{i}\hat{j}} \neq \bar{Q}_{\hat{i}\hat{j}}$ then let

$$\tilde{Q}_{ij} = \begin{cases} \hat{Q}_{ij}, & i \neq \hat{i} \\ \bar{Q}_{ij}, & i = \hat{i} \end{cases},$$

and let $\tilde{P} = \hat{P}$. In this case, the lemma once again follows from the same arguments given at the end of Lemma 4.3.3.

Next, suppose that no indices as described above exist for either $\hat{P}$ or $\hat{Q}$. Since $(\hat{P}, \hat{Q})$ is not aligned there must be some column $\hat{j}$ satisfying $\sum_i \hat{P}_{i\hat{j}} \geq 2$. Further, let $\hat{i}_1$ and $\hat{i}_2$ satisfy $P_{\hat{i}_1\hat{j}} = P_{\hat{i}_2\hat{j}} = 1$. At least one of these rows does not match its corresponding row in $\bar{P}$, so assume without loss of generality that $\hat{P}_{\hat{i}_1\hat{j}} \neq \bar{P}_{\hat{i}_1\hat{j}} = 0$. By assumption, $\hat{Q}_{\hat{j}\hat{i}_1} = 1$ and

$\hat{P}_{\hat{i}_2\hat{j}} = \overline{P}_{\hat{i}_2\hat{j}} = 1$. Thus, let

$$\tilde{Q}_{ij} = \begin{cases} \hat{Q}_{ij}, & i \neq \hat{i}_2 \\ \overline{Q}_{ij}, & i = \hat{i}_2 \end{cases},$$

and let $\tilde{P} = \hat{P}$ so that the lemma follows from the same arguments given at the end of Lemma 3. ■.

Based on Lemma 4.3.1, Lemma 4.5.3, and Lemma 4.5.4 it is clear that for $R_i \in \mathcal{O}$ we have $\gamma(R_i) = (N-1)(|\mathcal{O}|-1)+(J-|\mathcal{O}|)$ for N even and $\gamma(R_i) = N(|\mathcal{O}|-1)+(J-|\mathcal{O}|)$ for N odd. To construct the R_i-tree, simply exploit the fact that from any state we can reach a different state with minimal resistance that is closer to R_i in the sense of Hamming distance. On the other hand, for $R_i \notin \mathcal{O}$ the same three lemmas provide that $\gamma(R_i) \geq (N-1)(|\mathcal{O}|)+(J-|\mathcal{O}|-1)$ for N even and $\gamma(R_i) \geq N(|\mathcal{O}|) + (J - |\mathcal{O}| - 1)$ for n odd, completing the proof. ■

A consistent feature of all of the models we have thus far considered is the completeness of the communication graph. That is, players attempt to communicate with the entire population irrespective of how effectively they can do so. A consequence of this assumption is that all of these models predict that monomorphic language states are nearly always observed. The next section suggests a model where fitness depends only on communication within linguistic communities.

4.6 Linguistic communities

Stochastic evolutionary dynamics achieve maximum efficiency in atomic signaling games over the long run. Why does diversity persist in the modern language landscape despite clear benefits to linguistic universality? Many researchers do in fact take the models quite seriously and argue that observed diversity is a transient phenomenon— a relic of a bygone era of isolation. Models of language competition [63] consider differential equation models that are closely related to the replicator dynamics in the non-atomic signaling game and reach very much the same conclusions, painting a bleak picture for any sort of persistent diversity.

Exogenous factors such as geographic isolation [64] and language's role as an in-group marker [66] have been suggested to account for this discrepancy. More recently, researchers have proposed homophily, the tendency to associate with similar others, as a mechanism to account for the persistence of diverse linguistic communities [7]. However, the model views languages as abstract feature vectors. In this framework, the desirability of a language is based solely on similarity with neighbours. In contrast, signaling games model communication systems that can possess homonymy and synonymy, so that languages have varying degrees of intrinsic ambiguity. However, linguistic communities are not modelled.

We attempt to explain the observed persistence of diversity by suggesting an augmentation of the model that introduces linguistic community structure to agents' interactions. Our model is inspired by a model of opinion formation [71]. In that model each agents' opinion is a real number, and at each time step each agent updates her opinion to be the average of the opinions that differ from her own by at most a fixed threshold. Agents consider opinions that differ by more than the threshold to be unreasonable. Similarly, our agents define their linguistic community to be those other agents with whom they can communicate above a certain threshold. At each time step a randomly selected agent updates to the language within her community that achieves the highest utility in communication restricted to that same community. Intuitively, such a model ought to be friendly to diversity because disparate languages can coexist in different communities.

We are able to characterize the stochastically stable states of this model for a restricted set of parameters. In this case we find that only monomorphic language states are stochastically stable. However, this analysis is relevant only when stochastic shocks are extremely rare. Disruptive events with profound implications for the language landscape, such as the reintroduction of modern Hebrew in the 20th century, would seem to betray such assumptions. Simulation results are provided for higher levels of randomizing behavior, which suggest a strong tendency towards the formation of distinct linguistic communities. Linguistic coherence is high within these communities, but not between them. Alternatively, we make recourse to convergence rates. It has been shown in a closely related setting that systems like our own may require time to convergence that is exponential in the population size [68]. However, such systems may linger in metastable states over the medium-run [72], [73]. We present simulation results that suggest efficient but distinct communities can persist in this manner even when they should be expected to vanish over the long-run. A final simulation study shows that the threshold parameter defining the linguistic community structure has substantial effects on the relative sizes of the communities observed.

4.6.1 The model

At each time t we select a player i according to F, as before. This agent's neighbourhood

$$\begin{aligned} h_i(\mathbf{P}[t-1], \mathbf{Q}[t-1]) \equiv \{ j \neq i : \operatorname{trace}(P_i[t-1]Q_j[t-1] + P_j[t]Q_i[t-1]) > r\} \\ \cup \{ j \neq i : (P_j[t-1], Q_j[t-1]) = (P_i[t-1], Q_i[t-1])\}, \end{aligned}$$

is precisely the agents with whom she can communicate at a level above some fixed threshold $r \in (0, 2\min\{m,n\})$ along with those sharing her language. If her neighborhood is empty, i.e. $|h_i(\mathbf{P}[t-1], \mathbf{Q}[t-1])| = 0$, she picks a new language at random uniformly. Otherwise, she updates her language as

$$(P_i[t-1], Q_i[t-1]) = \begin{cases} (P_{\hat{k}}[t-1], Q_{\hat{k}}[t-1]), & \text{with probability } 1-\epsilon \\ \operatorname{rand}(\mathcal{L}_{m,n}), & \text{with probability } \epsilon \end{cases}.$$

where

$$\hat{k} \in \arg \max_{j \in h_i(\mathbf{P}[t], \mathbf{Q}[t])} \sum_{k \in h_i(\mathbf{P}[t], \mathbf{Q}[t])} \operatorname{trace}(P_j[t]Q_k[t]) + \operatorname{trace}(P_k[t]Q_j[t]).$$

All other agents continue with their previous language and a new agent is selected for revision as above.

Different values of m, n, and r will give a range of possible network structures. As an example, we illustrate the structure for $m = n = 2$, $r = 3$ in Fig. 21. An agent's neighbours are the agents who share her language or use any of the languages that her language is

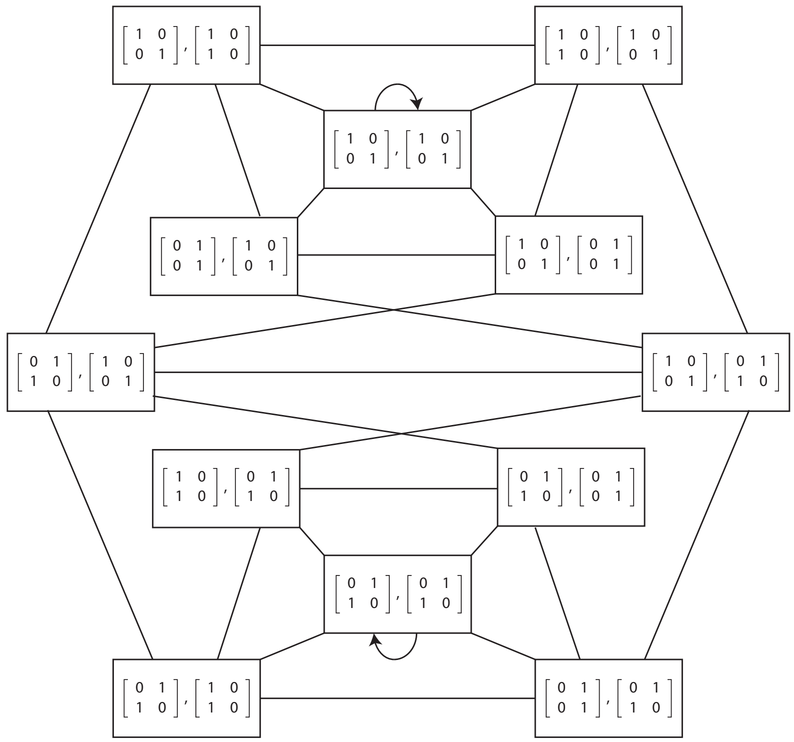

Figure 21: The network structure for $m = n = 2$, $r = 3$.

linked to in this graph. Notice that for these parameters only 12 of 16 languages possess links, and only two are linked to themselves.

The model has a somewhat contrived appearance because an agent that can communicate effectively within her community will with high probability eschew the opportunity to revise her language in order to communicate effectively with a larger community. We are altogether ignoring the advantages of incumbency that models of language competition concentrate on. It turns out that for small ϵ, even this is not enough. We use $\mathcal{P}^{\epsilon}_{m,n,r}$ to refer to the regular perturbed Markov process induced by the linguistic community model.

Theorem 4.6.1 *Let $m = n$ and $r \in (2(n-1), 2n)$. Then the set of stochastically stable states of $\mathcal{P}^{\epsilon}_{m,m,r}$ is precisely $\mathcal{O}$.*

As usual we will show that for any $x, y \in \mathcal{O}$ we have

$$\gamma(y) = \gamma(x) = \lambda(x) \equiv \gamma_{\mathcal{O}}. \tag{24}$$

We then establish stochastic stability of $\mathcal{O}$ by showing that for any $x \notin \mathcal{O}$ we have $\gamma(x) \geq \lambda(x) > \gamma_{\mathcal{O}}$.

From here on we refer to recurrent communication classes as just recurrent classes for brevity. Unlike in the models considered above, not all recurrent classes are absorbing states. Our proof will not characterize the recurrent classes aside from a few key features.

The unperturbed process $\mathcal{P}^0_{m,n,r}$ is not innovative (i.e. new languages never appear), so the set of languages in each state in a recurrent class is the same. More formally, consider any two states $z = (\mathbf{P}, \mathbf{Q})$ and $\hat{z} = (\hat{\mathbf{P}}, \hat{\mathbf{Q}})$ contained in a single recurrent class R_i. For any j there must exist k such that $(\hat{P}_k, \hat{Q}_k) = (P_j, Q_j)$. While each state in a recurrent class must have the same set of, the actual number of players speaking each language can vary from state to state. However, the number of players speaking each aligned language is constant across all of the states in the recurrent class. This is because agents speaking aligned languages never change languages so long as they have neighbours, a claim we establish with the following lemma.

Lemma 4.6.1 *Suppose that the state* $(\boldsymbol{P}, \boldsymbol{Q})$ *contains an aligned language* (P_i, Q_i) *with* $|h_i(\boldsymbol{P}, \boldsymbol{Q})| \geq 1$ *then each*

$$\hat{i} \in \arg\max_{j \in h_i(\boldsymbol{P},\boldsymbol{Q})} \sum_{k \in h_i(\boldsymbol{P},\boldsymbol{Q})} \operatorname{trace}(P_j Q_k + P_k Q_j), \tag{25}$$

satisfies $(P_{\hat{i}}, Q_{\hat{i}}) = (P_i, Q_i)$.

Proof: By the definition of the neighbourhood $h_i(\mathbf{P}, \mathbf{Q})$ we have

$$\operatorname{trace}(P_i Q_j + P_j Q_i) \geq 2n - 1, \tag{26}$$

for each $j \in h_i(\mathbf{P}, \mathbf{Q})$. Neighbours of an agent speaking an aligned language either speak the same aligned language or possess a zero column. If agent $j \in h_i(\mathbf{P}, \mathbf{Q})$ possess a zero column then for any $k \in h_i(\mathbf{P}, \mathbf{Q})$ we have

$$\operatorname{trace}(P_k Q_j + P_j Q_k) \leq 2n - 1, \tag{27}$$

and $\text{trace}(P_j Q_j + P_j Q_j) \leq 2n - 2$, so (P_i, Q_i) outperforms (P_j, Q_j) against all members of $h_i(\mathbf{P}, \mathbf{Q})$ and does so strictly against (P_j, Q_j). ■

Since the number of agents speaking any aligned language is non-decreasing for any state trajectory in the recurrent class, all states in each recurrent class must have the same number of agents speaking each aligned language.

Consider recurrent classes containing one or more aligned languages. A probability ϵ event is sufficient to move one agent from a language that is not aligned to one of the aligned languages present in the recurrent class. It does not matter which state we apply the perturbation from. This new state may be transient, but we are guaranteed to reach a new recurrent class with strictly more agents speaking aligned languages. This fact follows from the preceding lemma. Proceeding like this we can reach an absorbing state in which all agents speak aligned languages. Agents can then switch from one aligned language to another via probability ϵ events so that we reach a state in O and we required only transitions between absorbing states having resistance equal to one. Recall that redundancies are immaterial.

Next, consider a recurrent class R_k without any aligned languages. In this case each state $(\mathbf{P}, \mathbf{Q}) \in R_k$ contains some agent i achieving

$$\text{trace}(P_i Q_i) = \max_j \text{trace}(P_j Q_j) \equiv c(R_k). \tag{28}$$

We can reach a state that increases this quantity by one via a single ϵ probability event. Further, this can be done in such a manner that the agent speaking the new language will have a non-empty neighbourhood.

Lemma 4.6.2 *Suppose* $\text{trace}(PQ) \leq n - 1$. *Then there exists another language* $(\hat{P}, \hat{Q})$ *satisfying*

$$\text{trace}(\hat{P}\hat{Q}) = \text{trace}(PQ) + 1, \tag{29}$$

and

$$\text{trace}(\hat{P}Q + P\hat{Q}) \geq r = 2n - 1. \tag{30}$$

Proof: Assume that P has no zero columns and let $\hat{Q} = P'$. The proof when P has a zero column is symmetric. Let

$$C = \{j : \sum_k P_{jk} Q_{kj} \geq 1\}, \tag{31}$$

the set of indices of columns of Q that do not contribute to trace(PQ). The matrix Q has at most one zero column. First, suppose Q has a zero column and let

$$C_1 = \{j \in C : \sum_k Q_{kj} > 0\}, \tag{32}$$

the set of indices in C of non-zero columns of Q. Next, let

$$\hat{P}_{ij} = \begin{cases} 1, & i \in C_1, j = \min\{\operatorname{argmax}_k Q_{ki}\} \\ 0, & i \in C_1, j \neq \min\{\operatorname{argmax}_k Q_{ki}\} \\ P_{ij}, & \text{otherwise.} \end{cases} \tag{33}$$

This way, each row of $\hat{P}$ in C_1 is confined to the support of the corresponding column in Q. Thus,

$$\operatorname{trace}(\hat{P}Q) = \sum_{i \in C_1} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C - C_1} \sum_j \hat{P}_{ij} Q_{ji} \tag{34}$$

$$\geq \sum_{i \in C_1} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} \tag{35}$$

$$= \sum_{i \in C_1} \sum_{j = \min\{\operatorname{argmax}_k Q_{ki}\}} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} \tag{36}$$

$$= |C_1| + \sum_{i \in C^c} \sum_j P_{ij} Q_{ji} = |C_1| + |C^c| = n - 1, \tag{37}$$

which combined with trace($P\hat{Q}$) = trace(PP') = m establishes the first part of the lemma. The second part of the lemma is verified by computing

$$\operatorname{trace}(\hat{P}\hat{Q}) = \sum_{i \in C_1} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C - C_1} \sum_j \hat{P}_{ij} \hat{Q}_{ji} \tag{38}$$

$$\geq \sum_{i \in C^c} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C - C_1} \sum_j \hat{P}_{ij} \hat{Q}_{ji} \tag{39}$$

$$= \sum_{i \in C^c} \sum_j P_{ij} P_{ij} + \sum_{i \in C - C_1} \sum_j P_{ij} P_{ij} \tag{40}$$

$$= |C^c| + |C - C_1| = \operatorname{trace}(PQ) + 1. \tag{41}$$

To complete the proof of the lemma we instead suppose that Q has no zero column. We define the set $C_0 \subset C$ so that $j \in C_0 \Rightarrow \sum_k Q_{kj} > 0$ and $|C_0| = |C_1| - 1$. Put another way, C_0 is any set obtained by removing any one column index from C_1. Next, let

$$\hat{P}_{ij} = \begin{cases} Q_{ji}, & i \in C_0 \\ P_{ij}, & \text{otherwise,} \end{cases} \tag{42}$$

giving,

$$\text{trace}(\hat{P}Q) = \sum_{i \in C_0} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C - C_0} \sum_j \hat{P}_{ij} Q_{ji} \tag{43}$$

$$\geq \sum_{i \in C_0} \sum_j \hat{P}_{ij} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} \tag{44}$$

$$= \sum_{i \in C_0} \sum_j Q_{ji} Q_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} Q_{ji} \tag{45}$$

$$= |C_0| + \sum_{i \in C^c} \sum_j P_{ij} Q_{ji} = |C_0| + |C^c| = n - 1, \tag{46}$$

as required. Lastly,

$$\text{trace}(\hat{P}\hat{Q}) = \sum_{i \in C_0} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C^c} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C - C_0} \sum_j \hat{P}_{ij} \hat{Q}_{ji} \tag{47}$$

$$\geq \sum_{i \in C^c} \sum_j \hat{P}_{ij} \hat{Q}_{ji} + \sum_{i \in C - C_0} \sum_j \hat{P}_{ij} \hat{Q}_{ji} \tag{48}$$

$$= \sum_{i \in C^c} \sum_j P_{ij} P_{ij} + \sum_{i \in C - C_0} \sum_j P_{ij} P_{ij} \tag{49}$$

$$= |C^c| + |C - C_0| = \text{trace}(PQ) + 1, \tag{50}$$

completing the proof. ■

While the new state just described may be transient, we will eventually reach a recurrent class containing the new language. This is because the last speaker of this new language never abandons her language so long as she has a neighbour.

Lemma 4.6.3 *Suppose that the state* $(\boldsymbol{P}, \boldsymbol{Q})$ *contains a language* (P_i, Q_i) *with* $|h_i(\boldsymbol{P}, \boldsymbol{Q})| \geq 1$ *such that for all* $j \neq i$ *we have* $(P_j, Q_j) \neq (P_i, Q_i)$*. Further suppose*

$$\text{trace}(P_i Q_i) > \max_{j \neq i} \text{trace}(P_j Q_j), \tag{51}$$

then

$$i = \arg\max_{j\in h_i(\boldsymbol{P},\boldsymbol{Q})} \sum_{k\in h_i(\boldsymbol{P},\boldsymbol{Q})} \mathrm{trace}(P_j Q_k + P_k Q_j). \tag{52}$$

Proof: By the definition of the neighbourhood $h_i(\mathbf{P}, \mathbf{Q})$ and the uniqueness of (P_i, Q_i) we have $\mathrm{trace}(P_i Q_j + P_j Q_i) \geq 2n-1$ for each $j \in h_i(\mathbf{P}, \mathbf{Q})$. Each other language $j \in h_i(\mathbf{P}, \mathbf{Q})$ has $\mathrm{trace}(P_j Q_j) < \mathrm{trace}(P_i Q_i) \leq n$ so it achieves a payoff of at most $2n - 2$ against itself, while (P_i, Q_i) achieves at least $2n - 1$. Since (P_i, Q_i) outperforms each language strictly against at least one other language in the neighbourhood (namely, the language itself), it need only match that language against all other languages. Thus it is sufficient to show that for any two agents $k, j \in h_i(\mathbf{P}, \mathbf{Q})$ with $k \neq j$ we have $\mathrm{trace}(P_k Q_j + P_j Q_k) \leq 2n - 1$. Assume the contrary, i.e. that there exist two agents k and j with $k \neq j$ satisfying $\mathrm{trace}(P_k Q_j + P_j Q_k) = 2n$. This requires $P_k = Q'_j$ and $P_j = Q'_k$. Now since j is in $h_i(\mathbf{P}, \mathbf{Q})$ we know that either $P_j = Q'_i$ or $Q_j = P'_i$ because one of the trace terms must equal n. We will deal with the former case only since the latter will then follow from symmetry. By the same reasoning we have that either $P_k = Q'_i$ or $Q_k = P'_i$. If $P_k = Q'_i$ then

$$n = \mathrm{trace}(P_j Q_k) = \mathrm{trace}(P_k Q_k) < n, \tag{53}$$

a contradiction. If $Q_k = P'_i$ then

$$n = \mathrm{trace}(P_j Q_k) = \mathrm{trace}(Q'_i P'_i) = \mathrm{trace}(P_i Q_i). \tag{54}$$

If $\mathrm{trace}(P_i Q_i) = n$ then all its neighbours possess a zero column, so that at least one of the requirements $P_k = Q'_j$ or $P_j = Q'_k$ will violate row stochasticity. ■

Of course, we must guarantee that she continues to have a neighbour on the way to the recurrent class. The next lemma establishes that the last of the neighbors of the agent speaking the new language never abandons her language. That is, unless she abandons her language for the new language. This can happen only if the new language is aligned, but in that case we have reached the scenario described above and are done.

Lemma 4.6.4 *Suppose that the state* $(\boldsymbol{P}, \boldsymbol{Q})$ *contains a language* (P_i, Q_i) *with* $|h_i(\boldsymbol{P}, \boldsymbol{Q})| \geq 1$ *such that for all* $j \neq i$ *we have* $(P_j, Q_j) \neq (P_i, Q_i)$*. Further suppose*

$$\text{trace}(P_i Q_i) \leq n - 1, \tag{55}$$

then either (i),

$$i = \arg \max_{j \in h_i(\boldsymbol{P},\boldsymbol{Q})} \sum_{k \in h_i(\boldsymbol{P},\boldsymbol{Q})} \text{trace}(P_j Q_k + P_k Q_j), \tag{56}$$

or (ii), there exists $(\hat{P}, \hat{Q})$ *such that for all*

$$\hat{i} \in \arg \max_{j \in h_i(\boldsymbol{P},\boldsymbol{Q})} \sum_{k \in h_i(\boldsymbol{P},\boldsymbol{Q})} \text{trace}(P_j Q_k + P_k Q_j), \tag{57}$$

$(P_{\hat{i}}, Q_{\hat{i}}) = (\hat{P}, \hat{Q})$ *and* $\text{trace}(\hat{P}\hat{Q}) = n$*.*

Proof: We know that for each $j \in h_i(\mathbf{P}, \mathbf{Q})$ we have

$$\text{trace}(P_i Q_j + P_j Q_i) \geq 2n - 1. \tag{58}$$

Consider any two agents $k, j \in h_i(\mathbf{P}, \mathbf{Q})$ and assume

$$\text{trace}(P_k Q_j + P_j Q_k) = 2n. \tag{59}$$

If this is not possible then (i) obtains. Supposing it is possible we have $P_k = Q'_j$ and $P_j = Q'_k$. Now since j is in $h_i(\mathbf{P}, \mathbf{Q})$ we know that either $P_j = Q'_i$ or $Q_j = P'_i$ because one of the trace terms must equal n. We will deal with the former case only since the latter will then follow from symmetry. By the same reasoning we have that either $P_k = Q'_i$ or $Q_k = P'_i$. If $Q_k = P'_i$ then

$$n = \text{trace}(P_j Q_k) = \text{trace}(Q'_i P'_i) = \text{trace}(P_i Q_i), \tag{60}$$

a contradiction. Thus $P_k = Q'_i$ so that

$$n = \text{trace}(P_j Q_k) = \text{trace}(P_k Q_k), \tag{61}$$

so that if (i) does not obtain, then (ii) obtains because only aligned languages can outperform (P_i, Q_i) against its own neighbours. ■

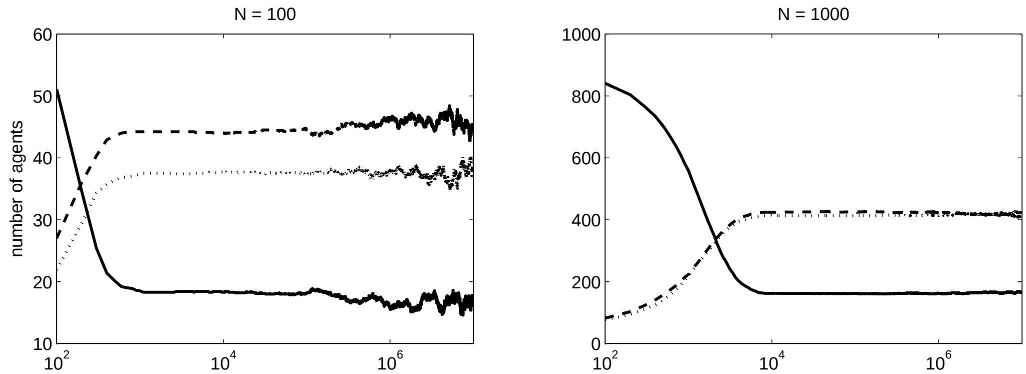

Figure 22: Stability of diversity varies with N; $r = 3, m = n = 2, \epsilon = 10^{-4}$, average of 10 runs.

We can apply the above method inductively so that $c(R_k)$ increases by one for each recurrent class R_k visited. We eventually reach a recurrent class containing an aligned language, from which point we have already established the existence of a suitable path to a state in $\mathcal{O}$. From these states in $\mathcal{O}$ all departing edges have resistance at least two. For a resistance tree rooted at a state in $x \in \mathcal{O}$, consider $y \in \mathcal{O}, y \neq x$. From any such state y we can move two players to the aligned language in x, giving a new absorbing state that achieves the minimum resistance from y of two. Then, we can move one player at a time to the aligned language in x, achieving a resistance of one for each absorbing state on our way to x. It follows that

$$\gamma(x) = \mu(x) = \gamma_{\mathcal{O}} = 2(|\mathcal{O}| - 1) + |\mathcal{O}^c| \tag{62}$$

For any other recurrent class $y \notin \mathcal{O}$ it is sufficient to note that any resistance tree has one more edge emanating from a state in $\mathcal{O}$, so that

$$\gamma(y) \geq \mu(y) = 2|\mathcal{O}| + |\mathcal{O}^c| - 1 > \gamma_{\mathcal{O}}, \tag{63}$$

completing the proof. ■

4.7 Simulations

Recall that the stochastically stable states are almost all we will see in the long-run. For these parameters, the elaborate linguistic community structure has thus made no difference

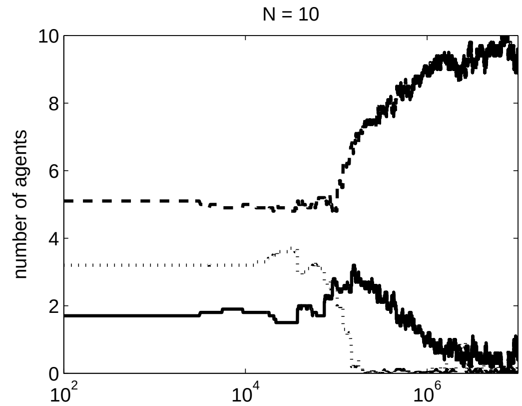

Figure 23: Distinct communities thrive over the medium-run; $r = 3, m = n = 2, \epsilon = 10^{-4}$, average of 10 runs.

at all from the viewpoint of stochastic stability . Our analysis does not preclude diversity over the medium-run or for larger values of ϵ, perspectives we now take up.

Stochastic stability characterizes long-run behavior, but such predictions may only become relevant after extraordinary lengths of time. Under more reasonable timescales states that are not stochastically stable may "appear" stable, a phenomenon referred to as metastability. Simulation results illustrated in the leftmost plot of Fig. 23 indicate this can occur for parameter values covered by Theorem 4.6.1. Two languages satisfy $\text{trace}(PQ) = m = n = 2$, these are the *aligned* languages and are represented with dotted lines. The wider dots indicates the more prevalent of the two[8]. The solid line sums over all other languages. Despite eventually settling into monomorphic states, the simulations indicate a metastable epoch where diverse communities thrive.

When ϵ is not small relative to the population size, we observe diverse, efficient linguistic communities even over long time-horizons . The impact of increasing the population size with ϵ fixed is illustrated in Fig. 22. For $\epsilon = 10^{-4}$ we eventually observe monomorphic states for small N, consistent with stochastic stability analysis. As N grows we observe two equally sized internally efficient linguistic communities[9]. Interestingly, the relative

[8]The identity of the more prevalent language is allowed to change. Either of the two states that is monomorphic in an aligned language can begin to dominate so, absent our convention, averaging over many runs would give the misleading appearance of diversity for large t

[9]Readers should exercise caution in drawing conclusions from these simulations when N is large because convergence rates may increase with N. However, even simulations initiated from monomorphic states (not shown) behaved similarly

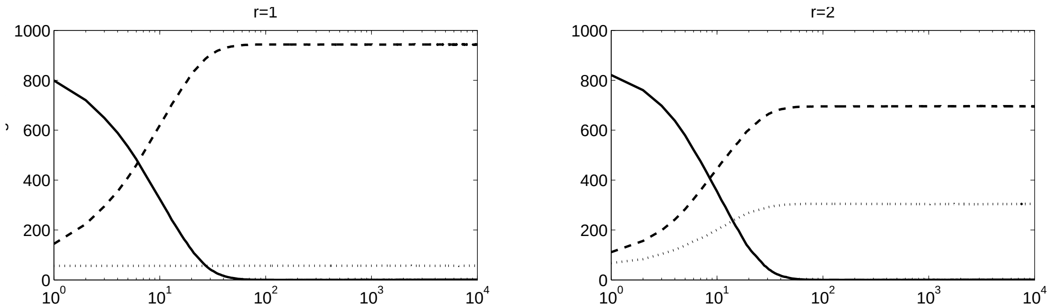

Figure 24: Community sizes vary with r; $m = n = 2, \epsilon = 10^{-4}, N = 1000$, average of 10 runs.

community sizes are related to the particular choice of threshold as illustrated in Fig. 24.

4.8 Proof of Lemma 4.3.7

With slight abuse of notation we define the *element-wise Hamming distance* between two matrices A and B as

$$d_H(A, B) = \frac{1}{2} \sum_i \sum_j |A_{ij} - B_{ij}|.$$

Let (P, Q) and $(\hat{P}, \hat{Q})$ be, respectively, the languages that R_i and R_j are monomorphic in. The basic idea of our argument is to construct a sequence of states per the requirements of Lemma 4.3.7 with P-matrices whose element-wise Hamming distances from $\hat{P}$ go to zero.

Suppose that we have a sequence of $L \geq 1$ language(s)

$$(P, Q) \equiv (P^1, Q^1), (P^2, Q^2), ..., (P^L, Q^L),$$

whose corresponding monomorphic states[10] satisfy all the requirements of Lemma 4.3.7 other than the second, so the terminal state is not $(\hat{\mathbf{P}}, \hat{\mathbf{Q}})$. If $P^L = \hat{P}$ then let $(P^{L+1}, Q^{L+1}) \equiv (P^L, \hat{Q}) = (\hat{P}, \hat{Q})$, so once again the lemma follows from the same argument given at the end of Lemma 4.3.3. It follows that we need only concentrate on driving down the distance between the P-matrices, since we can then trivially complete the sequence.

Let $\mathcal{K}(P^L) = \{(i, j) : P^L_{ij} > \hat{P}_{ij}\}$ and let

$$\hat{\mathcal{K}}(P^L) = \{(i, j) \in \mathcal{K}(P^L) : \sum_k P^L_{kj} \geq 2\}. \tag{64}$$

[10]For the remainder of the proof, when we make reference to sequences of this form, we will assume it is understood we are referring to the corresponding monomorphic states.

Suppose that $|\hat{\mathcal{K}}(P^L)| \geq 1$ and let

$$(i^*, j^*) \in \operatorname*{argmin}_{(i,j)\in\hat{K}(P^L)} \sum_k Q_{ki}.$$

Further, suppose $\sum_k Q^L_{ki^*} = 0$. In this case, since the i^*th column of Q^L is all zeros, we can modify the i^*th row of P^L, with the transition to the new monomorphic state having resistance equal to one, per Lemma 4.3.5. In particular, let

$$P^{L+1}_{ij} = \begin{cases} P^L_{ij}, & i \neq i^* \\ \hat{P}_{ij}, & i = i^* \end{cases}, \tag{65}$$

so that the sequence

$$(P, Q) \equiv (P^1, Q^1), (P^2, Q^2), ..., (P^L, Q^L), (P^{L+1}, Q^L) \tag{66}$$

satisfies all the requirements of Lemma 4.3.7 other than the second, but $d_H(P^{L+1}, \hat{P}) = d_H(P^L, \hat{P}) - 1$. Suppose instead that $\sum_k Q^L_{ki^*} = 1$. In this case, none of the mismatched rows in P^L correspond to zero columns in Q^L. The situation can be remedied as follows. Let $\hat{\imath} \neq i^*$ satisfy $P^L_{\hat{\imath}j^*} = 1$, which is achievable due to (64), and define

$$Q^{L+1}_{ij} = \begin{cases} Q^L_{ij}, & i \neq j^* \\ 0, & i = j^*, j \neq \hat{\imath} \\ 1, & i = j^*, j = \hat{\imath} \end{cases},$$

so that the sequence

$$(P, Q) \equiv (P^1, Q^1), (P^2, Q^2), ..., (P^L, Q^L), (P^L, Q^{L+1})$$

satisfies all the requirements of Lemma 7 other than the second. Obviously, this addition does not make any progress as far as $d_H(P^L, \hat{P})$ is concerned. However, we now have $\sum_k Q^L_{ki^*} = 0$ so that the sequence can be extended in the manner of (65) and (66).

We have thus far established that the sequence can be extended with $d_H(P^{L+1}, \hat{P}) = d_H(P^L, \hat{P}) - 1$ so long as $|\hat{\mathcal{K}}(P^L)| \geq 1$. Next, suppose $|\hat{\mathcal{K}}(P^L)| = 0$. In this case we may actually augment the sequence in such a manner that $d_H(P^{L+1}, \hat{P}) = d_H(P^L, \hat{P}) + 1$, in effect moving us further from our goal. However, we will be able to guarantee that subsequent steps decrease this distance by at least two, more than offsetting the increase. First, let j^* satisfy $\sum_k Q^L_{kj^*} = 0$, let $(\hat{i}, \hat{j}) \in \mathcal{K}$, define

$$\hat{P}^{L+1}_{ij} \leftarrow \begin{cases} P^L_{ij}, & i \neq j^* \\ 0, & i = j^*, j \neq \hat{j} \\ 1, & i = j^*, j = \hat{j} \end{cases},$$

and append (P^{L+1}, Q^L) to the sequence. Now, $d_H(P^{L+1}, \hat{P}) \geq d_H(P^L, \hat{P})$, because the row of P^L that we modify, j^*, already matches $\hat{P}$. To see this, assume the contrary so $(j^*, \tilde{j}) \in \mathcal{K}(P^L)$ for some $\tilde{j}$. We must also have $(j^*, \tilde{j}) \in \hat{\mathcal{K}}(P^L)$ because since (P^L, Q^L) is aligned we have $P^L_{i\tilde{j}} \geq Q^L_{\tilde{j}i}$ for all i, implying $\sum_k P^L_{k\tilde{j}} \geq 2$ because $Q^L_{\tilde{j}j^*} = 0$ by assumption, which contradicts our assumption that $|\hat{\mathcal{K}}(P^L)| = 0$.

Note that after this last addition we now have $(\hat{i}, \hat{j}) \in \hat{\mathcal{K}}(P^{L+1})$. Next, let $\tilde{i} \neq \hat{i}$ satisfy $\hat{P}_{\tilde{i}\hat{j}} = 1$, define

$$Q^{L+2} = \begin{cases} Q^L_{ij}, & i \neq \hat{j} \\ 0, & i = \hat{j}, j \neq \tilde{i} \\ 1, & i = \hat{j}, j = \tilde{i} \end{cases},$$

and append (P^{L+1}, Q^{L+2}) to the sequence so that $\sum_k Q^L_{k\hat{i}} = 0$. The sequence can now be extended extended in the manner of (65) and (66), so that

$$d_H(P^{L+3}, \hat{P}) = d_H(P^{L+1}, \hat{P}) - 1 \leq d_H(P^L, \hat{P}).$$

The final step is to show that subsequent steps enable a strict decrease in this distance when necessary, which follows once we show $|\hat{\mathcal{K}}(P^{L+3})| \geq 1$. To see this, first note that some column sum in P^{L+3} is greater than the corresponding column sum in P^{L+1}, but since P^{L+1} already had positive column sums (as (P^{L+1}, Q^{L+2}) is aligned), that column must sum to at least two. Consider $\mathcal{K}(P^L)$, the set of incorrect ones of P^{L+1}, whose column sums are equal to one by assumption. Let

$$\tilde{K}(P^L) = \{k : \exists(i, j) \in \mathcal{K}(P^L) \text{ s.t. } \hat{P}_{ik} = 1\},$$

the set of columns that the elements of $\mathcal{K}(P^L)$ should have ones in. For each $k \in \tilde{K}(P^L)$ there exists i such that $(i, k) \in \mathcal{K}(P^L)$, otherwise $\hat{P}$ would have a zero column, which is impossible due to $(\hat{P}, \hat{Q})$ being aligned. In P^{L+1} we increase one of these column sums to two so that in P^{L+3} a row can be corrected. Correcting a row moves a one to a column in $k \in \tilde{K}(P^L)$, but there are the same columns as $\mathcal{K}(P^L)$ so the column whose sum has increased to two cannot match $\hat{P}$. ■

CHAPTER 5

POPULATION GAMES AND PASSIVITY

This chapter begins the second half of this thesis. These results are more abstract than those described in the first half of the thesis, which were focused on the specific applications of self-assembly and language. Here we focus on the problem of distributed convergence to equilibrium more generally.

Stable games [15] have the attractive property of admitting global convergence to equilibria under many learning dynamics. We show that stable games can be formulated as passive input-output systems. This observation enables us to identify passivity of a learning dynamic as a sufficient condition for global convergence in stable games. Notably, dynamics satisfying our condition need not exhibit positive correlation between the payoffs and their directions of motion. We show that our condition is satisfied by the dynamics known to exhibit global convergence in stable games. We give a decision-theoretic interpretation for passive learning dynamics that mirrors the interpretation of stable games as strategic environments exhibiting self-defeating externalities. Moreover, we exploit the flexibility of the passivity condition to study the impact of applying various forecasting heuristics to the payoffs used in the learning process. Finally, we show how passivity can be used to identify strategic tendencies of the players that allow for convergence in the presence of information lags of arbitrary duration in some games.

5.1 Introduction

Among the oldest problems for game theory is the question of what, if anything, is the correct solution concept. The stock answer, Nash equilibrium, has well-documented difficulties. While existence is generally not an issue so long as we allow for mixed strategies, uniqueness can rarely be guaranteed. A common approach to identifying the "correct" prediction has been to analyze dynamic system models intended to mimic the decision making processes of the players. This bottom-up approach is often referred to as evolutionary game theory, or, learning in games. The procedure entails selecting an appropriate "evolutionary dynamic" (behavioral rule) and proceeding to analyze the action trajectories that the game induces, which often converge to Nash equilibria. The choice of dynamic is normally based on exogenous considerations. In particular, the most well-studied dynamic, the replicator dynamic [74], is inspired by biological evolution. For a thorough background on evolutionary game theory, the reader is advised to consult the many monographs on the subject. In particular, we will mostly follow the terminology and notation of [27].

Unfortunately, evolutionary game theory frequently fails to provide much additional clarity because oftentimes the evolutionary dynamic of interest for a game under study will not induce any stable fixed points, and may even exhibit chaos [75]. In fact, games have been constructed that can be shown to never exhibit stable Nash equilibria under only very mild conditions on the dynamics themselves [9]. Thus, from a worst-case perspective, evolutionary game theory has fundamental explanatory limitations. Nonetheless, the situation is often much better. In recent decades, researchers have sought to identify broad classes of games for which correspondingly broad classes of dynamics converge to equilibrium, most notably, potential games [12]. We focus here on the recently proposed notion of a stable game [15]— a generalization of a number of earlier ideas including concave potential games and symmetric normal form games with an interior ESS. The appealing property of stable games is that their Nash equilibria comprise a convex set that many dynamics are guaranteed to reach [15].

In this paper, we show that stable games can be formulated as passive input-output systems. Passivity is an abstraction of energy conservation and dissipation in mechanical and electrical systems [76] that has become a standard tool in the design and analysis of nonlinear systems [77], [78], [79]. It provides conditions under which particular system interconnections will be stable. After we identify stable games as passive systems, we are guaranteed that play by any admissible passive learning dynamic will admit globally asymptotically stable equilibria. It turns out that latter requirement is not especially restrictive as we show that the dynamics that guarantee global convergence in stable games are indeed passive.

Previous attempts at characterization of a broad class of learning dynamics achieving global convergence in stable games concentrated on the notion of positive correlation [15], that is, on the inner product between the payoffs and the direction of motion. Learning dynamics exhibiting global convergence in stable games can be shown to satisfy positive correlation, or in the case of perturbed best response dynamics, a variant termed virtual positive correlation [80]. However, a learning dynamic satisfying positive correlation that fails to converge in a stable game can be constructed, which motivated the additional requirement of integrability of the revision protocols. Nonetheless, positive correlation and integrability are not known to provide a sufficient condition for convergence in stable games— suitable Lyapunov functions must be identified for each learning dynamic independently. Although our sufficient condition for convergence, passivity, still requires us to find an analogous function, we find that the form of correlation required is between the time derivative of payoffs and the direction of motion. This correlation suggests an interpretation of passive learning that mirrors the interpretation of stable games as strategic environments exhibiting self-defeating externalities.

An immediate benefit of our characterization, beyond providing a sufficient condition for stability, is the novel generalizations it enables. Evolutionary game theory has historically placed particular emphasis on the study of memoryless, or “one-shot” games and the dynamical systems induced by play according to learning dynamics with order equal to the total number of strategies across all players. While our definitions include this setting, they are not restricted to it. Dynamic learning schemes that utilize additional, auxiliary states in reckoning strategy changes can also be analyzed using passivity. In particular, we identify games that preserve the convergence properties of passive learning dynamics when they are combined with prevalent forecasting heuristics like smoothing and trend following. Alternatively, certain dynamic games, that is, strategic enviornments where payoffs can depend on the entire action trajectory, can be shown to exhibit passivity.

Lastly, we probe the limits of the class of passive learning dynamics by suggesting a learning scheme in which players attempt to update strategies in a contrarian manner. Specifically, they discount payoffs to actions that have seen a rise in popularity over a defined lookback period. This scheme leads to an infinite-dimensional system. We find that this predisposition has no consequences for global convergence of passive dynamics in stable games and all other passive strategic environments. Furthermore, such behavior has the added benefit of preserving global convergence guarantees in some games even when otherwise destabilizing information lags are present.

Passivity techniques have been used in analysis of game theoretic learning dynamics employed in certain specific engineering models [81], [82], but the notion of passivity capturing a class of dynamics or games is novel as far as we know.

5.2 Background

5.2.1 Population Games and Evolutionary Dynamics

Let $\mathcal{P} = \{1, 2, ..., \mathfrak{p}\}$ be a *society* comprised of $\mathfrak{p} \geq 1$ *populations*. We think of each population p as a continuum of agents having *mass* m^p. We can informally think of an individual agent as an infinitesimal in one of the populations. Each population p has a set of available strategies $S^p = \{1, 2, ..., n^p\}$. The total number of strategies is denoted $n = \sum_{p\in\mathcal{P}} n^p$. The set of *strategy distributions* for population p is $X^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i\in S^p} x_i^p = m^p\}$ so that for $x^p \in X^p$ we understand $x_i^p \in \mathbb{R}_+$ as the mass of players in p utilizing strategy $i \in S^p$. The product $X = \prod_{p\in\mathcal{P}} X^p$ is the set of *social states*.

In this paper we will insist that the population masses remain constant. This implies that for $x^p, y^p \in X^p$ we have $\sum_{i\in S^p}(x^p - y^p) = 0$. Thus admissible changes in strategy are restricted to the *tangent space* $TX^p = \{z^p \in \mathbb{R}^{n^p} : \sum_{i\in S^p} z_i^p\}$. Similarly, changes in social state are confined to $TX = \prod_{p\in\mathcal{P}} TX^p$. We denote the orthogonal projection onto TX as Φ.

The payoff function $F : X \to \mathbb{R}$ is a continuous map associating each social state with a payoff vector so that $F_i^p : X \to \mathbb{R}^n$ is the payoff to strategy $i \in S^p$. We will often assume X to be fixed and refer to F itself as the game.

A state $x \in X$ is a *Nash equilibrium*, denoted $x \in NE(F)$ if each strategy in the support of x receives the maximum payoff available to its population, i.e.

$$x \in NE(F) \Leftrightarrow [x_i^p > 0 \Rightarrow F_i^p(x) \geq F_j^p(x)]$$

$$\forall i, j \in S^p \text{ and } p \in \mathcal{P}.$$

We next give a formal definition of deterministic evolutionary dynamics. Define sets $\mathcal{F}$ and $\mathcal{T}$ as follows:

$$\mathcal{F} = \{F : X \to \mathbb{R}^n : F \text{ is Lipschitz continuous}\};$$

$$\mathcal{T} = \{\{x_(t)\}_{t\geq 0} \subseteq X : x(\cdot) \text{ is continuous}\}.$$

A deterministic evolutionary dynamic is a set valued map $\mathbf{D} : \mathcal{F} \to \mathcal{T}$ that assigns each population game $F \in \mathcal{F}$ a set $\mathbf{D}(F) \subset \mathcal{T}$ such that for each $\zeta \in X$, there is a trajectory $\{x_t\}_{t\geq 0} \in \mathbf{D}(F)$ with $x_0 = \zeta$. We will give special attention to evolutionary dynamics specified by the initial value problem,

$$\dot{x} = V(x, F(x)) = V_F(x),$$

which we will call *traditional* learning dynamics.

5.2.2 Stable Games

We say that $F : X \to \mathbb{R}^n$ is a *stable game* if

$$(y - x)'(F(y) - F(x)) \leq 0 \quad \forall x, y \in X.$$

For a detailed discussion of stable games, see [15]. Many evolutionary dynamics are quite well-behaved when restricted to the stable games. The primary intent of this paper is to further formalize this observation. The above definition has an intuitive interpretation when F is continuously differentiable.

Theorem 5.2.1 *[15] Suppose the population game F is C^1, then F is a stable game if and only if $DF(x)$ is negative semidefinite with respect to TX for all $x \in X$.*

We say that such an F satisfies *self-defeating externalities*. That is, the payoff improvements to strategies being switched to are dominated by the payoff improvements to strategies being abandoned. This is easy to see by letting $z = e_j^p - e_i^p \in TX$, the difference between two unit vectors, and noting that (by definition) $z'DF(x)z \leq 0$. This implies that $\frac{\partial F_j^p(x)}{\partial z} \leq \frac{\partial F_i^p(x)}{\partial z}$, as required.

Many games are known to be stable games. For a thorough list consult [15] and [83]. We point out only a few important examples here.

Example 5.2.1 (Zero sum games) *Suppose that the number of populations*[1] $\mathfrak{p} = 2$ *and*

$$F(x^1, x^2) = \begin{bmatrix} 0 & A_1 \\ (A_2)' & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

If $A_2 = -A_1$ then we say that the game is zero sum. Indeed we have $\sum_{p \in \{1,2\}} \sum_{i \in S^p} F_i^p(x) = 0$ for all $x \in X$. If $z \in \mathbb{R}^n$, then

$$z'DF(x)z = (z^1)'A_1 z^2 - (z^2)'(A_1)'z^1 = 0,$$

implying F is a stable game. §

Example 5.2.2 (Network zero sum games[2]) *Consider a digraph G with $\mathcal{P}$ as its vertex set and a set of edges $E \subset \mathcal{P} \times \mathcal{P}$. Suppose that for each edge $(i, j) \in E$ there is a zero sum game $\left(A^{(i,j)}, A^{(j,i)}\right)$ played between the two populations i and j, where $A^{(j,i)} = -\left(A^{(i,j)}\right)'$. In the notation of the last example $A^{(i,j)}$ is A_1 and $A^{(j,i)} = A_2'$. The source plays the role of the first population and the sink plays the role of the second. We assume that payoffs are*

[1] In the context of zero sum games it is more common to interpret each population as a player that randomizes her actions according to her strategy distribution. We will use the population terminology throughout this paper.

arrived at by summation over subgames.

$$F(x) = \begin{bmatrix} 0 & A^{(1,2)} & A^{(1,3)} & \dots & A^{(1,\mathfrak{p})} \\ A^{(2,1)} & 0 & A^{(2,3)} & \dots & A^{(2,\mathfrak{p})} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{(\mathfrak{p},1)} & A^{(\mathfrak{p},2)} & A^{(\mathfrak{p},3)} & \dots & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^{\mathfrak{p}} \end{pmatrix},$$

where $A^{(i,j)} = 0$ if neither (i, j) or (j, i) is in E. For any $x \in X$ and any $z \in \mathbb{R}^n$ we have

$$\begin{aligned} z' DF(x) z &= \sum_{(i,j)\in E} \left((z^j)' A^{(i,j)} z^i + (z^i)' A^{(j,i)} z^j \right) \\ &= \sum_{(i,j)\in E} \left((z^j)' A^{(i,j)} z^i - (z^j)' A^{(i,j)} z^i \right) \\ &= 0, \end{aligned}$$

so F is a stable game. §

Example 5.2.3 (Concave potential games) *Suppose that $F : X \to \mathbb{R}^n$ satisfies $\nabla f = \Phi F$ for some function $f : X \to \mathbb{R}$. Then we call F a potential game and f its potential function. If, in addition, f is concave, then we say that F is a concave potential game. Concave potential games are stable games, as can be seen from*

$$\begin{aligned} (y - x)' (F(y) - F(x)) &= (\Phi(y - x))' (F(y) - F(x)) \\ &= (y - x)' (\Phi F(y) - \Phi F(x)) \\ &= (y - x)' (\nabla f(y) - \nabla f(x)) \\ &\leq 0, \end{aligned}$$

where the second equality follows from symmetry of the projection matrix Φ. §

Example 5.2.4 (Congestion games) *An important subclass of concave potential games are congestion games with increasing costs. These games model allocation of resources among selfish users with inelastic demand. Later on we will present some new results for this specific class, so we review the definitions here. Congestion games were originally* *proposed in [85] (also see [18], [86]). The form we study is that described in [87]. We begin with a finite set Ψ of facilities. For each $p \in \mathcal{P}$ there is a set S^p of subsets of Ψ—these are the strategies available to users in that population. Thus for each facility $\phi \in \Psi$ we define the utilization level*

$$u_\phi(x) = \sum_{p\in\mathcal{P}} \sum_{s\in S^p} a_{s,\phi} x_s^p,$$

where $a_{s,\phi}$ is the consumption rate of users of strategy s with respect to facility ϕ. Each facility has a non-decreasing cost function $c_\phi : \mathbb{R}_+ \to \mathbb{R}$. The payoff function for strategy $s \in S^p$ is given by

$$F_s^p = -\sum_{\phi\in s} a_{s,\phi} c_\phi(u_\phi(x)).$$

The payoffs can be more compactly represented as

$$F(x) = -U'C(Ux),$$

where $U \in \mathbb{R}^{|\Psi|\times n}$ is a utilization matrix satisfying $(Ux)_\phi = u_\phi(x)$ and

$$C\begin{pmatrix} u_1 \\ \vdots \\ u_{|\Psi|} \end{pmatrix} = diag\left(c_1(u_1), ..., c_{|\Psi|}(u_{|\Psi|})\right).$$

It follows that

$$DF(x) = -U'DC(Ux)U \leq 0, \quad \forall x \in X$$

because

$$DC(Ux) = diag\left(\frac{\partial c_1}{\partial u_1}(u_1(x)), ..., \frac{\partial c_{|\Psi|}}{\partial u_{|\Psi|}}(u_{|\Psi|}(x))\right) \geq 0,$$

since the costs are non-decreasing. The stable game property of congestion games can also be demonstrated by showing that congestion games are concave potential games. We provide this derivation in order to note that in the case that the cost functions are strictly increasing, $DF(x)$ is negative definite for almost all utilization matrices. However, we note that in some formulations the $a_{s,\phi}$ are assumed to all be equal to one, in which case there are only finitely many U and some of them may not be full rank. §

We will see that the stable games can be formulated as passive systems. First, we review classical passivity theory.

5.2.3 Passivity

We consider the input-output system

$$\dot{z} = f(z,u)$$
$$y = h(z,u)$$

where the function $f : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q$ is locally Lipschitz (i.e. Lipschitz when restricted to any compact set) and $h : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous. This system, given an initial state vector z_0, can be thought of as an operator from input trajectories $\{u_t\}_{t\geq 0} \subset \mathbb{R}^p$ to output trajectories $\{y_t\}_{t\geq 0} \subset \mathbb{R}^p$. We say the system is *passive* if there exists a continuously differentiable positive semidefinite function $L : \mathbb{R}^q \to \mathbb{R}_+$ (called the *storage function*) such that

$$u'y \geq \dot{L} = \frac{\partial L}{\partial z} f(z,u). \quad \forall (z,u) \in \mathbb{R}^q \times \mathbb{R}^p$$

For memoryless systems, we have $L = 0$. We assume for convenience that $z = 0$ is a fixed point of the dynamics with zero input, i.e. $f(0,0) = 0$. There is no loss of generality because any fixed point can be shifted to the origin with an appropriate change of variables.

Theorem 5.2.2 *If a system is passive with a positive definite storage function L then the origin of $\dot{z} = f(z,0)$ is stable.*

Proof: Take L as the Lyapunov function for $\dot{z} = f(x,0)$, then $\dot{L} = 0$. ■

In order to show asymptotic stability we must either show $\dot{L} < 0$ or apply LaSalle's invariance principle. We postpone further discussion of these issues. Now consider two passive systems, S_1 and S_2:

$$\dot{z}_i = f_i(z_i, u_i)$$
$$y_i = h_i(z_i, u_i), \quad i = \{1,2\}$$

where the f_i are locally Lipschitz and the h_i are continuous. We are interested in the form of interconnection shown in Fig. 1, that is:

$$e_1 = u_1 - h_2(z_2, e_2)$$

$$e_2 = u_2 + h_1(z_1, e_1).$$

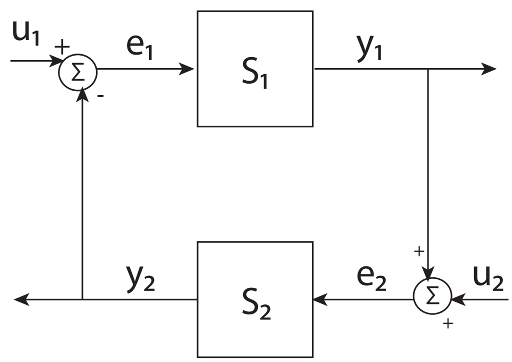

Figure 25: A negative feedback interconnection of two dynamic systems.

We will assume that these equations have unique solutions for every (z_1, z_2, u_1, u_2). This implies an overall state model:

$$\dot{x} = f(z, u)$$

$$y = h(z, u)$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The functions f and h inherit the smoothness properties of f_i and h_i.

Theorem 5.2.3 *The feedback connection of two passive systems is passive.*

Proof: Let the systems have storage functions $L_i, i = 1, 2$. Then

$$e_i' y_i \geq \dot{L}_i$$

From the interconnection we get

$$e_1'y_1 + e_2'y_2 = (u_1 - y_2)'y_1 + (u_2 + y_1)'y_2 = u_1'y_1 + u_2'y_2$$

This implies

$$u'y = u_1'y_1 + u_2'y_2 \geq \dot{L}_1 + \dot{L}_2$$

Let $L(z) = L_1(z_1) + L_2(z_2)$ so that

$$u'y \geq \dot{L}$$

as required. ■

We will need to modify these definitions a bit before they will be useful in the context of population games. We would like to, on the one hand, relax these definitions so they need only apply for a restricted set of admissible inputs, while at the same time constraining interconnections so that they guarantee our trajectories do not generate social states outside of X. Furthermore, it will be the positive interconnection (as opposed to negative) that we are interested in, which will require an appropriate notion of anti-passivity.

5.3 Main Results

We can represent a C^1 population game F as an input-output system,

$$\dot{x} = u$$

$$\dot{\pi} = DF(x)u$$

$$y = \dot{\pi} = DF(x)u,$$

which we refer to as the *game subsystem induced by F*. Traditionally, we think of games as memoryless mappings from strategy x to payoff $F(x)$. This alternative description is for mathematical convenience. We will thus think of games as mappings from strategy *trajectories* to payoff trajectories. Given an initial condition, x_0, any admissible, differentiable trajectory $\dot{x}$ can be "fed" into the game as an input u. By admissibility we refer to the requirement that

$$x(t) = x \in_0^t \dot{x}(\tau)d\tau + x_0 \in X, \ \forall t \geq 0.$$

The output $\dot{\pi}$ is then just the instantaneous time derivative of payoff. The actual payoff can be recovered by integrating and adding in the initial condition:

$$\pi(t) = \int_0^t \dot{\pi}(\tau)d\tau + \pi_0.$$

In order to extend passivity to a system of this form we need to precisely define the admissible inputs and similarly define passivity relative to those inputs.

5.3.1 $\mathcal{M}$-passivity

We will see that stable games and certain learning dynamics exhibit a form of passivity for systems with compact state spaces. Let

$$\mathcal{U} = C([0,\infty), \mathbb{R}^p),$$

the set of continuous functions mapping $[0,\infty)$ to $\mathbb{R}^p$. We start with $\mathcal{M} \subset \mathbb{R}^n$ and define an input space

$$\mathcal{U}_{\mathcal{M}}(v) \triangleq \{u \in \mathcal{U} : z_0 = v \Rightarrow z(t) \in \mathcal{M},\ \forall\, t \geq 0\}.$$

These are the inputs that keep the state of the system in $\mathcal{M}$ when the system is initialized at $z_0 = v$. Suppose that $\mathcal{U}_{\mathcal{M}}(v) \neq \{\emptyset\}$ for all $v \in \mathcal{M}$. Let $\mathcal{A} \subset \mathcal{M}$ be closed and let $\mathcal{Y} \subset \mathcal{M}$ be a neighborhood of $\mathcal{A}$. Further, suppose that there exists a continuous function $L : \mathcal{Y} \to \mathbb{R}_+$ with $L^{-1}(0) = \mathcal{A}$ such that

$$\dot{L}(z(t)) \leq \sigma u(t)'y(t),\ \forall\, z(t) \in \mathcal{Y} \text{ and } \forall\, u \in \mathcal{U}_{\mathcal{M}}(z_0), \tag{67}$$

where "$\cdot$" refers to the right upper Dini derivative, namely

$$\dot{L}(z(t)) = \limsup_{h\to 0^+} \frac{1}{h}\left(L\left(z(t+h)\right) - L\left(z(t)\right)\right).$$

Then we say that the system is $\mathcal{M}$-*passive* if $\sigma = 1$ and $\mathcal{M}$-*anti-passive* if $\sigma = -1$. If the inequality (75) is strict for $z(t) \notin \mathcal{A}$ then the system is *strictly* $\mathcal{M}$-passive if $\sigma = 1$ and *strictly* $\mathcal{M}$-anti-passive if $\sigma = -1$. The following theorem extends Theorem 5.2.2 to our setting.

Theorem 5.3.1 *Suppose a system is $\mathcal{M}$-passive or $\mathcal{M}$-anti-passive with storage function L and $0 \in \mathcal{U}_{\mathcal{M}}(v)$ for all $v \in \mathcal{M}$. Consider solutions to the autonomous system $\dot{z} = f(z, 0)$. If given any bounded set $\Omega \subset \mathcal{Y}$ there exists a bounded set $\Gamma(\Omega)$ such that $z_0 \in \Omega$ implies $z(t) \in \Gamma(\Omega)$ for all $t \geq 0$ then $L^{-1}(0)$ is Lyapunov stable. If, in addition to the above, the system is strictly $\mathcal{M}$-passive or strictly $\mathcal{M}$-anti-passive then $L^{-1}(0)$ is asymptotically stable, and globally asymptotically stable if $\mathcal{Y} = \mathcal{M}$.*

The proof of 5.3.1 uses standard Lyapunov arguments pertaining to stability of compact sets, such as those found in [88]. Since Theorem 5.3.1 is a special case of Theorem 5.9.1 below, we prove only the more general result. The existence of the compact set $\Gamma(\Omega)$ is nearly immediate when dealing with traditional game and dynamics subsystems. We will show how this requirement can be dispensed with below.

Consider the *positive*-feedback interconnection of two systems that are, respectively, $\mathcal{M}_1$-anti-passive wih storage function L_1 and $\mathcal{M}_2$-passive with storage function L_2. More precisely, we set

$$e_1 = -u_1 + h_2(z_2, e_2)$$
$$e_2 = u_2 + h_1(z_1, e_1).$$

In order to extend $\mathcal{M}$-passivity to the interconnection in a meaningful way, we must identify a non-empty set $\mathcal{M}_{12} \subset \mathcal{M}_1 \times \mathcal{M}_2$ so that, for the overall system, $0 \in \mathcal{U}_{\mathcal{M}_{12}}(v)$ for all $v \in \mathcal{M}_{12}$.

Theorem 5.3.2 *The positive-feedback interconnection of two systems that are, respectively, $\mathcal{M}_1$-passive with storage function L_1 and $\mathcal{M}_2$-passive with storage function L_2, is $\mathcal{M}_{12}$-passive with storage function $L_{12}(z_1, z_2) = L_1(z_1) + L_2(z_2)$.*

The proof is nearly identical to that of the classical passivity theorem. Theorem 5.3.1 can then be used to establish stability properties, the key point being recognition of $L_{12}(z_1, z_2)$

as the appropriate Lyapunov function. The next section illustrates how our definitions accommodate stable games.

5.4 Passive systems induced by games

Recall the game subsystem induced by F,

$$\dot{x} = u$$

$$\dot{\pi} = DF(x)u$$

$$y = \dot{\pi} = DF(x)u.$$

Theorem 5.4.1 *Stable games are X-anti-passive with storage function* 0.

Proof: Since

$$\dot{\pi} = DF(x)\dot{x},$$

we have that

$$\dot{\pi}'\dot{x} = \dot{x}'(DF(x))'\dot{x} \leq 0,$$

which implies

$$0 \leq -\dot{\pi}'\dot{x} = -u'y.$$

■ We note here that $DF(x)$ is not defined for social states $x \notin X$. It follows that the only allowed inputs $u = \dot{x}$ are those that that keep x in X. This is captured in the definition of X-anti-passivity.

5.5 Passive Dynamics

We can similarly view an evolutionary dynamic $\dot{x} = V_F(x)$ in this manner

$$\dot{\pi} = u$$

$$\dot{x} = V_\pi(x)$$

$$y = \dot{x} = V_\pi(x)$$

where $\dot{x}$ is understood as the time-derivative of the social state x **and** the definition of the system output. The social state x appears in both the learning and game subsystems. Abusing notation slightly, we use the subscripted variables $x_{\mathfrak{G}}$ and $x_{\mathfrak{D}}$ to distinguish the two. It is easy to see that the positive interconnection of the input-output systems associated with the game and learning dynamics recovers exactly the traditional differential equation as long as we initialize

$$x_{\mathfrak{G}}(0) = x_{\mathfrak{D}}(0) = x_0,$$

and

$$\pi(0) = F(x_0),$$

In particular plugging in for the inputs u in the learning dynamics and game equations gives

$$\dot{\pi} = DF(x)V_{\pi}(x)$$
$$\dot{x} = V_{\pi}(x),$$

but since $\pi(0) = F(x_0)$ this implies $\pi(t) = F(x(t))$ for all t so that we have simply

$$\dot{x} = V_F(x),$$

as required. However, we have transformed an ordinary differential equation system of order n to an interconnection of a system having order n with a system having order $2n$. We reiterate that for C^1 games and traditional learning dynamics this formulation is equivalent to the traditional one under the natural initialization. We will refer to the procedure just utilized as the natural dimensional reduction.

We seek passive (i.e. $\mathbb{R}^n \times X$-passive) dynamics that produce outputs $\dot{x} \in \mathcal{U}_X(x_0)$. We do not need to restrict the inputs ($\dot{\pi}$) to the dynamics in any way. The restriction $\dot{x} \in \mathcal{U}_X(x_0)$ is needed to ensure that interconnection with the game is meaningful, i.e. that we do not produce outputs that, when input to the game subsystem, produce motion out of X. We consider dynamics specified by *revision protocols*

$$\rho^p : \mathbb{R}^{n^p} \times X^p \to \mathbb{R}_+^{n^p \times n^p},$$

so that

$$\dot{x}_i^p = \sum_{j \in S^p} x_j^p \rho_{ji}^p(\pi^p, x^p) - x_i^p \sum_{j \in S^p} \rho_{ij}^p(\pi^p, x^p).$$

Thus we can regard $\rho_{ij}^p(\pi^p, x^p)$ as the switch rate from strategy $i \in S^p$ to strategy $j \in S^p$.

Our first example is the excess payoff target (EPT) dynamics. EPT dynamics have revision protocols of the form

$$\rho_{ij}^p(\pi^p, x^p) = \tau_j^p(\hat{\pi}^p),$$

where

$$\hat{\pi}^p = \pi^p - \frac{1}{m^p}\Big((x^p)'\pi^p\Big) \cdot \mathbf{1},$$

is the vector of *excess payoffs*. In this case the dynamics take the simpler form

$$\dot{x}^p = m^p \tau^p(\hat{\pi}^p) - \Big(\mathbf{1}'\tau^p(\hat{\pi}^p)\Big) \cdot x^p.$$

The EPT dynamics include best response[3], logit, and Brown-von Neumann-Nash dynamics. Clearly EPT dynamics always guarantee $\dot{x} \in \mathcal{U}_X(x_0)$. We say that τ^p is *separable* if $\tau_i^p(\hat{\pi}^p)$ is independent of $\hat{\pi}_{-i}^p$, and *acute* if

$$\tau^p(\hat{\pi}^p)'\hat{\pi}^p > 0 \text{ whenever } \hat{\pi}^p \in \mathbb{R}^{n_p} - \mathbb{R}_-^{n_p}.$$

We assume that τ^p is Lipschitz continuous. The prototype of this subset of the EPT dynamics is the Brown-von Neumann-Nash dynamic, with $\tau_i^p(\hat{\pi}_i^p) = [\hat{\pi}_i^p]_+$.

In the theorem that follows, we treat EPT dynamics as input-output systems from $\dot{\pi}$ to $\dot{x}$, as explained above. The proof is mostly reproduced from [15].

Theorem 5.5.1 *Seperable, acute EPT dynamics are strictly $\mathbb{R}^n \times X$-passive with storage function*

$$L_{EPT}(x, \pi) = \sum_{p \in \mathcal{P}} m^p \sum_{i \in S^p} \int_0^{\hat{\pi}_i^p} \tau_i^p(s) ds.$$

[3]Strictly speaking, best response dynamics are specified by a differential inclusion, we do not discuss these here.

Proof: It can be verified that $\mathtt{sgn}(\tau_i(\pi_i)) = \mathtt{sgn}([\pi]_+)$, so that each integral in L_{EPT} and hence V_{EPT} itself is non-negative. The zero level-set can be represented as

$$\begin{aligned}
L_{\text{EPT}}^{-1}(0) &= \left\{(x,\pi) : \int_0^{\hat{\pi}_i^p} \tau_i^p(s)ds = 0 \quad \forall i \in S^p, p \in \mathcal{P}\right\} \\
&= \{(x,\pi) : \hat{\pi}_i^p \leq 0 \quad \forall i \in S^p, p \in \mathcal{P}\} \\
&= \{(x,\pi) : \pi - (x'\pi) \cdot \mathbf{1} \in \mathbb{R}_-^n\} \\
&= \{(x,\pi) : x_i^p > 0 \Rightarrow \pi_i^p = \max_{j \in S^p} \pi_j^p\} \\
&= \{(x,\pi) : x \in NE(\pi)\} \\
&\neq \{\emptyset\}.
\end{aligned}$$

The last equality indicates that x is a strategy distribution that is Nash for the constant payoff game π. Note that this has little to do with the traditional notion of Nash equilibria because we have not imposed any game structure. We proceed to verify the passivity inequality.

$$\begin{aligned}
\dot{L}_{\text{EPT}} &= \sum_{p \in \mathcal{P}} m^p (\tau^p(\hat{\pi}^p))' \frac{d}{dt}\Big(\pi^p - \frac{1}{m^p}((x^p)'\pi^p) \cdot \mathbf{1}\Big) \\
&= \sum_{p \in \mathcal{P}} m^p (\tau^p(\hat{\pi}^p))' \Big(\dot{\pi}^p - \frac{1}{m^p} \frac{d}{dt}((x^p)'\pi^p) \cdot \mathbf{1}\Big) \\
&= \sum_{p \in \mathcal{P}} (\dot{x}^p + \mathbf{1}'\tau^p(\hat{\pi}^p) \cdot x^p)' \Big(\dot{\pi}^p - \frac{1}{m^p} \frac{d}{dt}((x^p)'\pi^p) \cdot \mathbf{1}\Big).
\end{aligned}$$

Multiply these terms and use that

$$\mathbf{1}'\dot{x}^p = 0 \quad \& \quad \mathbf{1}'x^p = m^p$$

to get

$$\begin{aligned}
\dot{L}_{\text{EPT}} &= \sum_{p \in \mathcal{P}} (\dot{x}^p)'\dot{\pi}^p + \mathbf{1}'\tau^p(\hat{\pi}^p)\Big((x^p)'\dot{\pi}^p - \frac{d}{dt}((x^p)'\pi^p)\Big) \\
&= \sum_{p \in \mathcal{P}} (\dot{x}^p)'\dot{\pi}^p + \mathbf{1}'\tau^p(\hat{\pi}^p)(- (\dot{x}^p)'\pi^p).
\end{aligned}$$

Therefore

$$\dot{x}'\dot{\pi} = \dot{L}_{\text{EPT}} + \sum_{p \in \mathcal{P}} \mathbf{1}'\tau^p(\hat{\pi}^p)((\dot{x}^p)'\pi^p).$$

Each summand in the second term in the RHS has the same sign as $(\dot{x}^p)'\pi^p$ or is zero because each $\tau_i^p \geq 0$. It was shown in [89] that acuteness of τ implies that $(\dot{x}^p)'\pi^p > 0$ whenever $\dot{x}^p \neq \mathbf{0}$ (positive correlation). Therefore

$$\dot{L}_{\text{EPT}} \leq \dot{x}'\dot{\pi} = u'y,$$

as required. We claim the inequality binds precisely on the set $L_{\text{EPT}}^{-1}(0)$. We have that $\dot{x} = 0$ on $L_{\text{EPT}}^{-1}(0)$ and $\dot{L}_{\text{EPT}}(x, \pi) = 0$ whenever $\dot{x} = 0$. Outside of $L_{\text{EPT}}^{-1}(0)$ we have $\mathbf{1}'\tau^p(\hat{\pi}^p) > 0$ and $(\dot{x}^p)'\pi^p > 0$, proving the claim. ■

The next section combines the results of the previous two sections to recover the known convergence properties of separable, acute EPT dynamics in stable games.

5.5.1 Interconnections

In order to realize traditional games/dynamics we restrict ourselves to initializations that assign the same strategy distributions to both the game and dynamics subsystems. The payoffs, residing only in the dynamics subsystem, are initialized so as to match the initial strategy distribution. Formally, we consider initializations from the invariant set

$$\mathcal{M}_{\mathcal{G},\mathcal{D}} = \{(x_{\mathcal{G}}, \pi, x_{\mathcal{D}}) : x_{\mathcal{G}} = x_{\mathcal{D}} \in X, \pi = F(x)\}.$$

We trivially have $0 \in \mathcal{U}_{\mathcal{M}_{\mathcal{G},\mathcal{D}}}(v)$ for all $v \in \mathcal{M}_{\mathcal{G},\mathcal{D}}$.

Combining the storage functions of stable games and separable, acute EPT dynamics restricted to the proposed invariant set and applying the natural dimensional reduction gives zero level set for the combined storage function

$$(L_{\text{EPT}} + 0)^{-1}(0) \cap \mathcal{M}_{\mathcal{G},\mathcal{D}} = NE(F),$$

corresponding to precisely the set of Nash equilibria of the stable game. Theorems 5.3.1 - 5.5.1 provide an alternative proof of the following known result, reworded below.

Theorem 5.5.2 *(Hofbauer and Sandholm [15]) The positive-feedback interconnection of a C^1 stable game F and separable, acute EPT dynamics admits the globally asymptotically stable set $NE(F)$.*

Proof: Verification of the boundedness requirement of Theorem 5.3.1 is the only step needed to prove the theorem, since strict $\mathcal{M}_{\mathcal{G},\mathcal{D}}$-passivity has already been established above. Since $\mathcal{M}_{\mathcal{G},\mathcal{D}} \subset X \times \mathbb{R}^n \times X$, π satisfy $\pi = F(x), x \in X$ and F is a continuous function over a compact set, boundedness follows. ■

The advantage of the present formulation is that the passivity inequality gives a sufficient condition for converge to equilibrium in stable games. In particular we see that positive correlation need not be satisfied. Instead, we check the correlation between the inner product of the time derivative terms $\dot{x}$ and $\dot{\pi}$. Indeed it had already been shown that positive correlation in an excess payoff dynamic not satsifying integrability could lead to cycling in some stable games [15]. In addition, the perturbed best response dynamics, a class of dynamics not satisfying positive correlation, do achieve global convergence in stable games [80]. We will illustrate passivity of perturbed best response dynamics, as well as a third family of dynamics known as pairwise comparison dynamics, below. First, we offer a decision theoretic interpretation of the passivity inequality for the learning subsystem. We borrow from the game theoretic interpretation of stable games.

5.5.2 Interpreting the passivity inequality

Inspection of $L_{\mathrm{EPT}}^{-1}(0)$ reveals the basic action of the EPT dynamics. In essence, EPT dynamics act as a greedy optimizer. If no exogenous input is supplied (i.e. $\dot{\pi} = 0$), then EPT dynamics will eventually ensure that only strategies enjoying maximum payoff among their population will be utilized. The passivity of EPT dynamics did not make any assumptions on $\dot{\pi}$. In particular, $\dot{\pi}$ need not be generated from interconnection with a game. We will exploit this property later by generalizing the game subsystem.

The passivity inequality for the EPT dynamics can be given an interpretation that borrows from the interpretation proposed for stable games. At $L_{\mathrm{EPT}}^{-1}(0)$, all populations are content. We can thus think of L_{EPT} as a metric of discontent, although obviously it is not a true metric. The passivity inequality says that the growth rate of agents' discontent is always less than the instantaneous self-enforcing externalities, $\dot{x}'\dot{\pi}$. These are the negative of self-defeating externalities. This guarantee is independent of the procedure generating $\dot{\pi}$. If the source of $\dot{\pi}$ is interconnection with a stable game, then instantaneous self-enforcing externalities are zero and so our metric of discontent does not grow at all!

To see this more clearly, consider an $\mathcal{M}$-passive system with storage function L. Suppose that there exists a strictly increasing function

$$\alpha : \mathbb{R}_+ \to \mathbb{R}_+$$

with $\alpha(0) = 0$ and

$$\alpha\left(\min_{\zeta \in A} \|z - \zeta\|\right) \leq L(z), \quad \forall z \in \mathcal{Y}.$$

We then have $\alpha^{-1}(L(z))$ as an upper bound on the distance from $L^{-1}(0)$, along with an upper bound on the growth rate of L. In this case, the boundedness requirements in Theorem 5.3.1 can be dispensed with. Since the payoff trajectory need not be confined to any compact set, we see that the passivity inequality provides guarantees even for scenarios that do not arise from interconnection with subsystems induced by games.

We show that such a function exists for acute, separable EPT dynamics and a single population with unit mass. The extension to multiple populations is straightforward. Let $\hat{k} = \operatorname{argmax}_k \hat{\pi}_k$. Then we can lower bound the largest excess payoff term

$$\begin{aligned}
\hat{\pi}_{\hat{k}} &= \pi_{\hat{k}} - \sum_{i \in S} x_i \pi_i \\
&= (1 - x_{\hat{k}})\pi_{\hat{k}} - \sum_{j \neq \hat{k}} x_j \pi_j \\
&= \sum_{j \neq \hat{k}} x_j (\pi_{\hat{k}} - \pi_j) \\
&\geq \sum_{j \neq \hat{k}} (\min\{x_j, \pi_{\hat{k}} - \pi_j\})^2 \\
&\geq \left(\min_{\zeta \in L_{\mathrm{EPT}}^{-1}(0)} \|(x, \pi) - \zeta\|\right)^2
\end{aligned}$$

Let $\mu(x, \pi) = \min_{\zeta \in L_{\mathrm{EPT}}^{-1}(0)} \|(x, \pi) - \zeta\|$, then

$$
\begin{aligned}
L_{\mathrm{EPT}}(x,\pi) &\geq \min_k \int_0^{\hat{\pi}_{\hat{k}}} \tau_k(s)ds \\
&\geq \min_k \int_0^{\mu^2(x,\pi)} \tau_k(s)ds.
\end{aligned}
$$

as required.

The notion of considering performance guarantees for evolutionary learning dynamics faced with an arbitrary stream of payoffs is also found in [90], where replicator dynamics are shown to possess the asymptotic no regret property, also see [91].

The next section establishes passivity of two additional families of learning dynamics: pairwise comparison dynamics and perturbed best response dynamics.

5.6 Other passive dynamics

5.6.1 Pairwise comparison dynamics

In this section we consider revision protocols satisfying

$$\rho_{ij}^p(\pi^p, x^p) = \phi_{ij}^p(\pi_j^p - \pi_i^p),$$

where the $\phi_{ij}^p : \mathbb{R} \to \mathbb{R}_+$ are Lipschitz continuous switch rates that depend only on pairwise comparison of payoffs. The prototype of this class is the Smith dynamic, with $\phi_{ij}^p = [\pi_j^p - \pi_i^p]_+$. In general, if the protocols ϕ^p satisfy *sign preservation*

$$\operatorname{sgn}(\phi_{ij}^p(\pi_j^p - \pi_i^p)) = \operatorname{sgn}([\pi_j^p - \pi_i^p]_+),$$

then they are called *pairwise comparison dynamics* [70]. If, in addition, the dynamics satisfy *impartiality*

$$\phi_{ij}^p(\pi_j^p - \pi_i^p) = \phi_j^p(\pi_j^p - \pi_i^p),$$

then they are passive.

Theorem 5.6.1 *Impartial pairwise comparison dynamics are strictly $\mathbb{R}^n \times X$-passive with storage function*

$$L_{PC}(x,\pi) = \sum_{p\in\mathcal{P}} \sum_{i\in S^p} \sum_{j\in S_p} x_i^p \int_0^{\pi_j^p - \pi_i^p} \phi_j^p(s)ds,$$

having zero level-set $\{(\pi, x) : x \in NE(\pi)\}$.

The proof of Theorem 5.6.1 is a straightforward extension of the proofs of Theorem 5.5.1 and the proof of Theorem 7.1 in [15]. Global asymptotic stability of $NE(F)$ upon interconnection with a stable game is then immediate, providing a slightly different procedure for arriving at Theorem 7.1 in [15]. The novelty of course is that pairwise comparison dynamics are stable upon interconnection with any passive subsystem, a property we will illustrate later.

Thus far we have considered only dynamics satisfying positive correlation. In the next section we consider perturbed best response dynamics, which do not satisfy positive correlation.

5.6.2 Perturbed best response dynamics

Like the other dynamics we have considered, perturbed best response dynamics admit globally asymptotically stable rest points under stable games. Unlike the other dynamics, perturbed best response dynamice do not satisfy positive correlation. While they do satisfy an analogue known as virtual positive correlation, this section identifies perturbed best response dynamics as yet another passive dynamic. Thus perturbed best response dynamics support the contention that the essential correlation is that implied by the passivity inequality. The analysis is again almost entirely borrowed, with the novel insight being the identification of passivity as the unifying property shared by dynamics known to guarantee global convergence in stable games.

A perturbed best response dynamics is the EPT dynamic obtained when τ^p is the *perturbed maximizer function* $\tilde{M}^p : \mathbb{R}^{n^p} \to \mathrm{int}(\Delta^p)$ given by

$$\tilde{M}^p(\pi^p) = \underset{y^p \in \mathrm{int}(\Delta^p)}{\mathrm{argmax}} (y^p)'\pi^p - v^p(y^p),$$

where $v^p : \mathrm{int}(\Delta^p) \to \mathbb{R}$ is differentiably strictly convex and infinitely steep at the boundary of Δ^p for each p[4]. As an example, the logit dynamics are obtained when v^p is the negated

[4]Functions with these properties are called *admissible deterministic perturbations* [80].

entropy function

$$v^p(y^p) = \eta \sum_{j\in S^p} y_j^p \log y_j^p.$$

Theorem 5.6.2 *Perturbed best response dynamics are strictly $\mathbb{R}^n \times X$-passive with storage function*

$$L_{PBR}(x,\pi) = \sum_{p\in\mathcal{P}} m^p \left(\tilde{\mu}^p(\hat{\pi}^p) + v^p\left(\frac{1}{m^p}x^p\right)\right),$$

where

$$\tilde{\mu}^p(\hat{\pi}^p) = \max_{y^p\in \mathrm{int}(\Delta^p)} \left((y^p)'\hat{\pi}^p - v^p(y^p)\right),$$

having zero level-set

$$\{(\pi, x) : x^p = m^p \operatorname*{argmax}_{y\in\mathrm{int}(\Delta^p)}(y'\pi^p - v^p(y))\}.$$

Proof: The storage function $L_{\mathrm{PBR}}(x,\pi)$ is clearly non-negative because each term in the summation is non-negative. Thus. we have $L_{\mathrm{PBR}}(x,\pi) = 0$

$$\begin{aligned}
&\Leftrightarrow \max_{y^p\in\mathrm{int}(\Delta^p)} \left((y^p)'\hat{\pi}^p - v^p(y^p)\right) + v^p\left(\frac{1}{m^p}x^p\right) = 0\\
&\Leftrightarrow \max_{y^p\in\mathrm{int}(\Delta^p)} \left((y^p)'\pi^p - v^p(y^p)\right) - \frac{1}{m^p}(x^p)'\pi^p + v^p\left(\frac{1}{m^p}x^p\right) = 0\\
&\Leftrightarrow \frac{1}{m^p}x^p \in \operatorname*{argmax}_{y\in\mathrm{int}(\Delta^p)}(y'\pi^p - v^p(y)).
\end{aligned}$$

That the set $\operatorname{argmax}_{y\in\mathrm{int}(\Delta^p)}(y'\pi^p - v^p(y))$ is a singleton follows from strict convexity of v^p, establishing that $L_{\mathrm{PBR}}^{-1}(0)$ is as described in the theorem. The passivity inequality then follows from the proof of Theorem 3.1 in [80] after replacing $F(x)$ with π. ■

Further analysis shows that perturbed best response dynamics converge to the unique perturbed equilibrium $PE(F)$ [80] in stable games. The perturbed equilibrium is a singleton that approximates $NE(F)$ when the perturbation is concentrated near the origin.

The next section suggests some new learning dynamics whose analysis is aided by our definitions.

5.7 Dynamic learning

In this section we consider learning dynamics that are less myopic than the standard examples we have considered thus far. In particular, we examine the consequences of players' utilizing either smoothed versions of the payoffs, or payoffs augmented with an additive term that approximates the time-derivative of the payoffs. Each of these modifications captures the application of some form of forecasting heuristic to the payoffs. In each case we show that for games having affine payoffs $F(x) = Ax + b$, with A negative definite, neither anticipation or smoothing have any consequences for global stability of passive learning dynamics. The prototype for this class of games is congestion games with affine, strictly increasing costs and non-singular utilization matrices. Later on we consider some generalizations of these heuristics.

A potential source of confusion is that we will be analyzing the properties of subsystems we had previously identified with the "game". This is for mathematical convenience. The dynamic learning rules are arrived at by applying a standard learning dynamic (e.g. EPT) to modified payoffs $\tilde{\pi}$. Our approach is to derive passivity results for modified "game" subsystems that map action trajectories to modified payoff trajectories. Interconnection with a passive learning dynamic then implies convergence results. Thus the problem of finding a class of games for which dynamic learning rules are well behaved can be cast as the problem of finding games that induce modified subsystems that are passive.

5.7.1 Smoothed learning

Suppose that $F(x) = Ax + b$, then we define the *smoothed learning subsystem induced by* F as

$$\dot{\pi} = Au$$
$$y = \dot{\tilde{\pi}} = \epsilon(\pi - \tilde{\pi}).$$

The term $\dot{x} = u$ is no longer needed because $DF(x) = A$ is now independent of F. The operational payoffs $\tilde{\pi}$ track the usual payoffs π, reflecting a state of affairs in which players utilize smoothed versions of a nominal payoff stream. Given a nominal payoff stream $\pi(t), t \geq 0$, the smoothed payoffs are given by

$$\tilde{\pi}_i(t) = e^{-\epsilon t}\tilde{\pi}(0) + \int_0^t e^{\epsilon(s-t)}\pi(s)ds. \tag{68}$$

The operational payoff $\tilde{\pi}$ is an exponentially weighted moving average of π with smoothing factor $\epsilon > 0$. These sort of moving averages are naive yet popular heuristics used to smooth out short term fluctuations in order to isolate longer term trends. Alternatively, smoothing may be unavoidable when the players can only process information subject to bandwitdth limitations.

For games with $A < 0$ we find that this form of payoff smoothing admits passive systems.

Theorem 5.7.1 *Let $F(x) = Ax + b$ with A negative definite. Then the smoothed learning subsystem induced by F is strictly $\mathbb{R}^{2n}$-anti-passive with storage function*

$$L_{s1}(\pi, \tilde{\pi}) = -\frac{\epsilon}{2}(\pi - \tilde{\pi})' A^{-1}(\pi - \tilde{\pi}),$$

having zero level set $\{(\pi, \tilde{\pi}) \in \mathbb{R}^{2n} : \pi = \tilde{\pi}\}$.

Proof: Clearly $L_{s1}^{-1}(0)$ is as described. We procede to verify the passivity inequality.

$$\begin{aligned}
\dot{L}_{s1}(\pi, \tilde{\pi}) &= -\epsilon(\pi - \tilde{\pi})' A^{-1}(\dot{\pi} - \dot{\tilde{\pi}}) \\
&= -\dot{\tilde{\pi}}'(\dot{x} - A^{-1}\dot{\tilde{\pi}}) \\
&= -\dot{\tilde{\pi}}'\dot{x} + \dot{\tilde{\pi}}' A^{-1}\dot{\tilde{\pi}} \\
&\leq -\dot{\tilde{\pi}}'\dot{x} = -u'y.
\end{aligned}$$

■

An immediate corollary of Theorem 5.7.1 is that any learning dynamic obtained from an admissible passive dynamic with smooth measurement of payoffs has a stable equilibrium. Indeed, for the standard dynamics we have studied global asymptotic stability of the applicable equilibrium set is guaranteed.

Theorem 5.7.2 *Let $F = Ax + b$ with $A < 0$ and consider the interconnection of the smoothed learning subsystem induced by F and either separable, acute EPT dynamics or impartial paiwise comparison dynamics. Then initializations from the invariant set*

$$\mathcal{M}_{\mathcal{G},\mathcal{D}} = \{(\pi_{\mathcal{G}}, \tilde{\pi}_{\mathcal{G}}, \pi_{\mathcal{D}}, x_{\mathcal{D}}) : \pi_{\mathcal{G}} = F(x_{\mathcal{D}}), \pi_{\mathcal{D}} = \tilde{\pi}_{\mathcal{G}}\},$$

admit, using the natural dimensional reduction, the globally asmyptotically stable set $NE(F)$.

Proof: We first check that the invariant set $\mathcal{M}_{\mathcal{G},\mathcal{D}}$, is the right one. It is straightforward to verify that the overall system reduces to the correct system

$$\dot{x} = V^{\mathtt{D}}_{\tilde{\pi}}(x)$$
$$\dot{\tilde{\pi}} = \epsilon(F(x) - \tilde{\pi}),$$

where $\mathtt{D} \in \{\mathtt{EPT}, \mathtt{PC}\}$. Also, $0 \in \mathcal{U}_{\mathcal{M}_{\mathcal{G},\mathcal{D}}}(v)$ for all $v \in \mathcal{M}_{\mathcal{G},\mathcal{D}}$, as required. Boundedness of the set of solutions follows from the arguments given in the proof of Theorem 5.5.2 and application of the mean value theorem to (68). It then follows from Theorems 5.3.1 - 5.5.1 that the set

$$(L_{\mathtt{D}} + L_{\mathrm{sp}})^{-1}(0) \cap \mathcal{M}_{\mathcal{G},\mathcal{D}}$$

is globally asymptotically stable. This set is equivalent to

$$\{(\pi_{\mathcal{G}}, \tilde{\pi}_{\mathcal{G}}, x_{\mathcal{D}}, \pi_{\mathcal{D}}) : \pi_{\mathcal{D}} = \tilde{\pi}_{\mathcal{G}} = \pi_{\mathcal{G}} = F(x_{\mathcal{D}}), x_{\mathcal{D}} \in NE(\pi_{\mathcal{D}})\},$$

which, under the natural dimensional reduction gives precisely the set $NE(F)$. ■

An analogous result can be developed for perturbed best response dynamics. To avoid redundancy, we will mostly avoid providing any more arguments of this form. Instead we will stop at demonstrating passivity of the subsystems we study, with the understanding that stability results can then be easily verified. We next consider anticipatory learning.

5.7.2 Anticipatory learning

Consider the following dynamic system, induced by an affine game $F(x) = Ax + b$.

$$\dot{\pi} = Au$$

$$\dot{\lambda} = \epsilon(\pi - \lambda)$$

$$y = \dot{\tilde{\pi}} = \dot{\pi} + \gamma\ddot{\lambda} = A(1 + \gamma\epsilon)u - \gamma\epsilon^2(\pi - \lambda).$$

We call this the *anticipatory learning subsystem induced by F*. The intention is that players respond to an augmented payoff $\tilde{\pi} = \pi + \gamma\omega$, where ω is an estimate of $\dot{\pi}$ and $\gamma > 0$ is the relative weight given to ω. In the system above, the quantity $\dot{\lambda}$ provides the estimate of $\dot{\pi}$ via an approximate differentiator. The fidelity of the approximation is controlled by $\epsilon > 0$, with larger values providing better estimates.

The concept of anticipatory learning was introduced in [92], and is inspired by classical methods in automatic control as well as the psychological tendency to extrapolate from past trends. In [92], players are able to observe their opponents' strategies and then use anticipatory learning to estimate their opponents' future strategy. The players then respond according to either fictitious play or gradient play. Here, we do not presume that players can observe their opponents' actions. The players use anticipatory learning to produce estimates of future payoffs. We study the stability properties of the overall system obtained when players respond to the augmented payoffs using passive dynamics.

For affine games with A negative definite, we find that anticipatory learning has no consequences for passive dynamics.

Theorem 5.7.3 *Let $F(x) = Ax + b$ with A negative definite. Then the anticipatory learning subsystem induced by F is strictly $\mathbb{R}^{2n}$-anti-passive with storage function*

$$L_{a1}(\pi, \lambda) = -\frac{\gamma^2\epsilon^3}{4 + 2\gamma\epsilon}(\pi - \lambda)'A^{-1}(\pi - \lambda),$$

having zero level set $\{(\pi, \lambda) \in \mathbb{R}^{2n} : \pi = \lambda\}$.

Proof: Clearly $L_{\mathtt{a1}}^{-1}(0)$ is as described, and closed. We procede to verify the passivity inequality. We have $\dot{L}_{\mathtt{a1}}(\pi, \lambda)$

$$\begin{aligned}
&= \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}(\pi-\lambda)'A^{-1}(\dot{\pi}-\dot{\lambda}) \\
&= \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}\left(\frac{(1+\gamma\epsilon)Au}{\gamma\epsilon^2} - \frac{y}{\gamma\epsilon^2}\right)' A^{-1}\left(\frac{-Au}{\gamma\epsilon} + \frac{y}{\gamma\epsilon}\right) \\
&= \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}u'A\frac{1+\gamma\epsilon}{\gamma\epsilon^2}A^{-1}A\frac{-1}{\gamma\epsilon}u - \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}u'A\frac{1+\gamma\epsilon}{\gamma\epsilon^2}A^{-1}\frac{1}{\gamma\epsilon}y \\
&\quad + \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}y'\frac{1}{\gamma\epsilon^2}A^{-1}A\frac{-1}{\gamma\epsilon}u + \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}y'\frac{1}{\gamma\epsilon^2}A^{-1}\frac{1}{\gamma\epsilon}y.
\end{aligned}$$

The first and fourth terms are negative due to $A < 0$, so combining the second and third terms gives

$$\begin{aligned}
\dot{L}_{\mathtt{a1}}(\pi, \lambda) &\leq \frac{-\gamma^2\epsilon^3}{2+\gamma\epsilon}u'\left(\frac{1+\gamma\epsilon}{\gamma^2\epsilon^3} + \frac{1}{\gamma^2\epsilon^3}\right)y \\
&= -u'y,
\end{aligned}$$

as required. ■

The storage function $L_{\mathtt{a1}}$ is valid for any $\gamma, \epsilon > 0$, so that passivity is guaranteed regardless of the weight given to the approximate derivative or the fidelity of the approximation.

The next two sections provide passivity results for more general forecasting heuristics.

5.7.3 Linear dynamic learning

For the cases of smooth and anticipatory learning we found explicit storage functions for subsystems induced by affine games with negative definite A matrices. In this section we consider the more general case of linear dynamic learning. Suppose that the augmented payoffs are arrived at by filtering the nominal payoffs (and their time derivatives) with a linear time-invariant system

$$\begin{aligned}
\dot{\omega} &= M_1\omega + M_2(\pi - \hat{\pi}) + B_1\dot{\pi} \\
\dot{\hat{\pi}} &= M_3\omega + M_4(\pi - \hat{\pi}) + B_2\dot{\pi}
\end{aligned}$$

or, equivalently, players react to modified payoffs

$$\hat{\pi}(t) = Ce^{Mt}\begin{bmatrix} \omega(0) \\ \hat{\pi}(0) \end{bmatrix} + C\int_0^t e^{M(t-s)}\begin{bmatrix} M_2 \\ M_4 \end{bmatrix}\pi(s)ds + C\int_0^t e^{M(t-s)}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\dot{\pi}(s)ds,$$

where $C = \begin{bmatrix} I_{n\times n} & 0_{n\times p} \end{bmatrix}$ and

$$M = \begin{bmatrix} M_1 & -M_2 \\ M_3 & -M_4 \end{bmatrix}.$$

Once again, $\hat{\pi}$ are the augmented payoffs that will serve as the input to the dynamics subsystem. The auxiliary states $\omega \in \mathbb{R}^p$ are "internal states" of the dynamic learning process and only impact decision making through $\hat{\pi}$. Anticipatory learning, smooth learning, standard learning (i.e. $\hat{\pi} = \pi$), and weighted averages of the three forms are all instances of this class of payoff augmentations. Recall that a payoff augmentation is a primitive that induces dynamic learning in the original game when interconnected with a learning subsystem. It is for mathematical convenience that we lump the payoff augmentation into the game subsystem.

The overall game subsystem is then given by

$$\begin{aligned}
\dot{\pi} &= Au \\
\dot{\omega} &= M_1\omega + M_2(\pi - \hat{\pi}) + B_1 Au \\
\dot{\hat{\pi}} &= M_3\omega + M_4(\pi - \hat{\pi}) + B_2 Au \\
y &= \dot{\hat{\pi}}.
\end{aligned}$$

Verifying if a system of this form is passive is challenging in general. However, because this system is linear time-invariant, passivity can often be verified using a computationally efficient procedure. Linear systems possessing the property of positive realness[5] are passive. A linear system is positive real if and only if an associated system of linear matrix inequalities (LMI) has a feasible point.

[5] We will not formally define positive realness here because we will be able to construct sufficient conditions for passivity and prove their correctness without doing so. Formal treatments can be found in many standard texts such as [79].

Due to the linearity of the system, if an appropriate storage function exists for the system then its domain can always be taken to be all of $\mathbb{R}^{2n+P}$. As with our earlier examples, we are concerned with stability properties of the Nash equilibria of the "one-shot" game. Therefore, we seek to verify passivity with respect to the equilibrium set $\{(\pi, \hat{\pi}, \omega) : \pi = \hat{\pi}, \omega = 0\}$. Establishing global convergence upon interconnection with a learning subsystem (e.g. impartial pairwise comparison dynamics) then follows from arguments that are nearly identical to those given in Theorem 5.7.2. The following theorem provides a sufficient condition for passivity of subsystems induced by linear dynamic learning payoff augmentations of affine games.

Theorem 5.7.4 *A system induced by linear dynamic learning in an affine game is $\mathbb{R}^{2n+p}$-anti-passive with respect to the equilibrium set $\{(\pi, \hat{\pi}, \omega) : \pi = \hat{\pi}, \omega = 0\}$ if there exists $P \geq 0$ satisfying*

$$\begin{bmatrix} P\hat{M} + \hat{M}'P & P\hat{B} + \hat{C}' \\ \hat{B}'P + \hat{C} & B_2 + B_2' \end{bmatrix} \leq 0, \tag{69}$$

where $\hat{B} = \begin{bmatrix} A - B_2 & B_1 \end{bmatrix}'$, $\hat{C} = \begin{bmatrix} M_4 & M_3 \end{bmatrix}$, and

$$\hat{M} = \begin{bmatrix} -M_4 & -M_3 \\ M_1 & M_2 \end{bmatrix}.$$

Moreover, if a strictly feasible solution to (69) exists then the system is strictly $\mathbb{R}^{2n+p}$-anti-passive.

Proof: First, we perform a change of coordinates that reduces the state space to $\mathbb{R}^{n+p}$ and shifts the equilibrium set to the origin. Let $\lambda = \pi - \hat{\pi}$ and let $z = \begin{bmatrix} \lambda & \omega \end{bmatrix}$, giving the system

$$\dot{z} = \hat{M}z + \hat{B}u$$

$$y = \hat{c}z + B_2 u,$$

which is equivalent to our original system in the sense that, given any input u, the two systems produce precisely the same output y. Suppose that the conditions of the theorem are satisfied. What remains is to show that the modified system is $\mathbb{R}^{n+p}$-anti-passive with respect to the origin. Use the storage function $L_{\mathtt{1dl}}(z) = \frac{1}{2}z'Pz$, so that $\dot{L}_{\mathtt{1dl}} + u'y$

$$
\begin{aligned}
&= z'P(\hat{M}z + \hat{B}u) + u'(\hat{C}z + \hat{D}u) \\
&= z'(P\hat{B} + \hat{C})u + \frac{1}{2}u'(B_2 + B_2')u + \frac{1}{2}z'(P\hat{M} + \hat{M}'P)z \\
&= \frac{1}{2}\begin{bmatrix} z \\ u \end{bmatrix}' \begin{bmatrix} P\hat{M} + \hat{M}'P & P\hat{B} + \hat{C}' \\ \hat{B}'P + \hat{C} & B_2 + B_2' \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \leq 0,
\end{aligned}
$$

as required. ■

The conditions of Theorem 5.7.4 are necesarry and sufficient conditions for positive realness, a property of the Fourier domain representation of the system. However, they are only sufficient conditions for passivity. The appeal of Theorem 5.7.4 is that it provides a condition that is efficiently checkable using widely available software packages.

The next section considers dynamic learning based on payoff augmentations that are nonlinear.

5.7.4 Nonlinear dynamic learning

Consider the following dynamic system, induced by an affine game $F(x) = Ax + b$,

$$
\begin{aligned}
\dot{\pi} &= Au \\
\dot{\hat{\pi}}_i &= \psi_i(\pi - \hat{\pi}) \quad i = 1, 2, ..., n \\
y &= \dot{\hat{\pi}},
\end{aligned}
$$

where for every i, $\psi_i : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, nondecreasing, and satisfies $\psi_i^{-1}(0) = \{0\}$. We furnish two examples.

Example 5.7.1 (Approximately constant-rate tracking) *Suppose that for each i*

$$\psi_i(y) = \begin{cases} -\beta_i, & y \leq -\alpha \\ \frac{\beta_i}{\alpha} y, & -\alpha < y \leq \alpha \\ \beta_i, & y \geq \alpha \end{cases}, \tag{70}$$

where each $\beta_i > 0$. Obseve that

$$\lim_{\alpha \to 0} \psi_i(y) = \beta_i \operatorname{sgn}(y),$$

so that for small α the ψ_i approximately correspond to players discretizing the time derivatives of the payoffs so that $|\dot{\pi}_i| \in \{0, \beta_i\}$ for each i. That is, they believe that when payoffs change they must do so at the constant rates given by the β_i. It is straightforward to constuct ψ_i that implement more finely grained approximate discretizations of the payoffs' rates of change, including asymetric discretizations where $\psi_i(y) \neq \psi_i(-y)$. §

Example 5.7.2 (Heterogeneous smoothing) *Suppose that $\psi_i(y) = \epsilon_i y$ for each i so that players perform smoothed learning, but with different smoothing factors for each strategy. Despite this case being an instance of linear dynamic learning, it was not covered by Theorem 5.7.1.*§

For a restricted class of games we find that the sometimes bizarre biases captured by nonlinear payoff augmentations have no consequences for global convergence when interconnected with passive learning dynamics.

Theorem 5.7.5 *Let $F(x) = Ax + b$ with $A = \operatorname{diag}(a_1, ..., a_n)$ negative definite. Then the system induced by F under nonlinear dynamic learning is strictly $\mathbb{R}^{2n}$-anti-passive with storage function*

$$L_{nd1}(\pi, \hat{\pi}) = -\sum_i a_i^{-1} \int_0^{\pi_i - \hat{\pi}_i} \psi_i(s) ds,$$

having zero level set $\{(\pi, \hat{\pi}) \in \mathbb{R}^{2n} : \pi = \hat{\pi}\}$.

Proof:

$$\begin{aligned}
\dot{L}_{\mathtt{ndl}} &= \sum_i (-a_i^{-1})\psi_i(\pi_i - \hat{\pi}_i)(\dot{\pi}_i - \dot{\hat{\pi}}_i) \\
&= \sum_i (-a_i^{-1})\dot{\hat{\pi}}_i(\dot{\pi}_i - \dot{\hat{\pi}}_i) \\
&= \sum_i -y_i u_i + a^{-1} y_i^2 \\
&= -u'y + y'A^{-1}y \leq -u'y.
\end{aligned}$$

■

In the next section we examine the consequences of dynamics in the assignment of payoffs.

5.8 Dynamic games

The interconnection of a passive earning dynamic with a subsystem induced by a stable game is stable. However, since any passive subsystem is stabilized by passive dynamics, we can identify subsystems not induced by games in the traditional sense that will also be stable under passive dynamics. In particular, we consider games with memory. That is, strategic environments where payoffs are allowed to depend on not just current play, but potentially the entire history of play. Both differential games and Markov games possess this feature. The equilibrium notions in those settings are more sophisticated than the "one-shot" equilibria studied in population games due to the more elaborate policy spaces that players are assumed to utilize. We instead study the stability of fixed points of dynamic games under myopic passive dynamics. This sort of mildly dynamic game environment is similar to the state based potential games [93]. In our framework, the game dynamics are a nuisance—players' strategic decisions are predicated on where actions and payoffs have been, not where they might go. The dynamic games we consider will again be induced by static games that are affine in the strategy distribution, that is $F(x) = Ax + b$.

5.8.1 Smooth measurement of actions

Consider the following dynamic system,

$$\dot{x} = u$$
$$\dot{\tilde{x}} = \epsilon(\tilde{x} - x)$$
$$y = \dot{\tilde{\pi}} = A\epsilon(\tilde{x} - x),$$

which we call the subsystem induced by F under smooth measurement of actions. The operational strategy distribution $\tilde{x} \in X$ is a smoothed version of the true strategy distribution x. The parameter $\epsilon > 0$ controls the weighting of actions in the more distant past relative to more recent ones. The payoffs awarded are based on the operational strategy distribution. This framework models smooth assignment of payoffs. That is, the game "remembers" where the strategy distribution was in the past and only slowly updates it's estimate of the strategy distribution used in assigning payoffs.

We leave it as a simple exercise for the reader to verify that if x and $\tilde{x}$ are each initialized in X, then $\tilde{x}$ remains in X as long as x does. We find that for a restriced class of games, smooth action measurement has no consequences for the stability of passive dynamics.

Theorem 5.8.1 *Let $F(x) = Ax + b$ with A symmetric and negative semidefinite with respect to TX. Then the system induced by F under smooth measurement of actions is $X \times X$-anti-passive with storage function*

$$L_{LAP}(x, \tilde{x}) = -\frac{\epsilon}{2}(x - \tilde{x})' A(x - \tilde{x}),$$

having zero level-set $\{(x, \tilde{x}) : x = \tilde{x}\}$. Moreover, if A is negative definite with respect to TX then the system is strictly $X \times X$-anti-passive.

Proof:

$$\begin{aligned}
\dot{L}_{\mathtt{smp}} &= -\epsilon(x - \tilde{x})' A(\dot{x} - \dot{\tilde{x}}) \\
&= -\dot{\tilde{\pi}}'(\dot{x} - \dot{\tilde{x}}) \\
&= -\dot{\tilde{\pi}}'\dot{x} + \dot{\tilde{x}}' A \dot{\tilde{x}} \\
&\leq -\dot{\tilde{\pi}}'\dot{x} = -u'y,
\end{aligned}$$

noting that the inequality is strict for $z \notin L_{\mathtt{smp}}^{-1}(0)$. ■

The proof of Theorem 5.8.1 is of course very similar to that of Theorem 5.7.1. We can similarly define dynamic games with anticipatory payoff assignment, as well as linear and non-linear generalizations in the manner of Theorem 5.7.4 and Theorem 5.7.5. In each case, demonstration of passivity is a straightforward extension of the arguments for the corresponding dynamic learning scheme

All of these dynamic learning schemes and dynamic games entail some form of memory of past payoffs or actions through the presence of auxiliary states in the system. In the next section we consider a more direct notion of memory. That is, games and dynamics whose evolution is allowed to depend explicitly on prior actions and states.

5.9 Time delays

In this section we propose methods allowing for global convergence to equilibrium in the presence of time delays, also referred to as information lags. The prospect of time delays destabilizing an interior ESS under replicator dynamics is studied in [94]. A similar analysis is carried out for discrete time replicator dynamics in [95]. The region of stability for replicator dynamics in the multiple access game and hawk-dove game under asymmetric delays is characterized in [96]. The emergence of cycling behavior in congestion games with binary choices under discrete time dynamics subject to time delays is studied in [97]. We begin our discussion with a brief review of the theory of retarded functional differential equations.

5.9.1 Retarded functional differential equations

Let $C([-T,0],\mathbb{R}^q)$ denote the set of continuous functions mapping the interval $[-T,0]$ into $\mathbb{R}^q$ where $T \geq 0$. For elements of $C([-T,0],\mathbb{R}^q)$ we define the norm

$$||z|| = \sup_{t\in[-T,0]} ||z(t)||$$

along with the associated Banach space

$$\overline{C}([-T,0],\mathbb{R}^q) = \{z \in C([-T,0],\mathbb{R}^q) : ||z|| \text{ is finite}\}.$$

Let D be a subset of $\mathbb{R} \times C([-T,0],\mathbb{R}^q)$, $f : D \to \mathbb{R}^q$, and let "$\cdot$" represent the right-hand derivative, then we call

$$\dot{z}(t) = f(t, z^t) \tag{71}$$

a *retarded functional differential equation* (RFDE) on D. This form of course includes ordinary differential equations. If $\Omega \subset D$ is open, $f : \Omega \to \mathbb{R}^q$ is continuous and $f(t,\phi)$ is Lipschitz continuous in ϕ when restricted to any compact subset of Ω then there is a unique solution on some interval to the initial value problem corresponding to (71) for any $(\sigma,\phi) \in \Omega$. This fact follows from the Schauder fixed point theorem, see [98]. Global existence of solutions however is not in general guaranteed and the existence of non-continuable solutions follows from Zorn's lemma. Appropriate Lyapunov arguments will be introduced later towards that end. When f depends on z^t evaluated at only finitely many points $-T \leq \tau_1, ..., \tau_l < 0$, many authors write simply

$$\dot{z}(t) = f(t, z(t), z(t-\tau_1), ..., z(t-\tau_l)), \tag{72}$$

to emphasize that the right-hand derivative depends on the usual state as well as a lagged version of that state, as opposed to the entire infinite-dimensional state $z^t \in C([-T,0],\mathbb{R}^q)$. As before in the finite-dimensional setting, we will suppose that the non-autonomous (i.e. time varying) component of the system is supplied by some exogenous input u so that

$$\dot{z}(t) = f(z^t, u(t)), \tag{73}$$

$$y(t) = h(z^t(0), u(t)) \tag{74}$$

where $u \in \mathcal{U}$ and the system output $h : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q$ are continuous. We will sometimes refer to $u : [0, \infty) \to C([-T, 0], \mathbb{R}^p)$, i.e. in functional form, in order to describe dependencies on lagged inputs. Lyapunov stability (and hence passivity) can be extended to this setting.

5.9.2 Passivity of RFDE's

The passivity definitions in this section will follow those in Section 5.3.1. As before, we will want to accommodate restrictions on the set of inputs as well as the possibility of equilibrium sets. We start with $\mathcal{M} \subset C([-T, 0], \mathbb{R}^q)$ and define an input space

$$\mathcal{U}_{\mathcal{M}}(v) \triangleq \{u \in \mathcal{U} : z^0 = v \Rightarrow z^t \in \mathcal{M},\ \forall\, t \geq 0\},$$

where $z^t, t \in [0, \infty)$ is a solution to the initial value problem (73) for $v \in \mathcal{M}$. These are the inputs that keep the state of the system in $\mathcal{M}$ when the system is initialized at $z^0 = v$. Suppose that $\mathcal{U}_{\mathcal{M}}(v) \neq \{\emptyset\}$ for all $v \in \mathcal{M}$. Let $\mathcal{A} \subset \mathcal{M}$ be closed and let $\mathcal{Y} \subset \mathcal{M}$ be a neighborhood of $\mathcal{A}$. Further, suppose that there exists a continuous function $L : \mathcal{Y} \to \mathbb{R}_+$ with $L^{-1}(0) = A$ such that

$$\dot{L}(z^t) \leq \sigma u(t)' y(t) - \delta \|y(t)\|^2,\ \forall\, z(t) \in \mathcal{Y} \tag{75}$$

and

$$\forall\, u \in \mathcal{U}_{\mathcal{M}}(z^0),$$

where $\delta \geq 0$, and "$\cdot$" refers to the right upper Dini derivative (assumed to be finite and measurable), namely

$$\dot{L}(z^t) = \limsup_{h \to 0^+} \frac{1}{h} \left(L(z^{t+h}) - L(z^t) \right).$$

Then we say that the system is $\mathcal{M}$-*passive* if $\sigma = 1$ and $\mathcal{M}$-*anti-passive* if $\sigma = -1$. If the inequality (75) is strict then the system is *strictly* $\mathcal{M}$-passive if $\sigma = 1$ and *strictly* $\mathcal{M}$-anti-passive if $\sigma = -1$. If $\delta \geq 1$ then we say the system is *output strictly* $\mathcal{M}$*-(anti)-passive.* Theorem 5.3.1 can be extended to this setting.

Theorem 5.9.1 *Suppose an RFDE system is $\mathcal{M}$-passive or $\mathcal{M}$-anti-passive with storage function L and $0 \in \mathcal{U}_{\mathcal{M}}(v)$ for all $v \in \mathcal{M}$ and consider solutions to the autonomous system $\dot{z} = f(z^t, 0)$. If given any bounded set $\Omega \subset Y$ there exists a compact set $\Gamma(\Omega)$ such that $z^0 \in \Omega$ implies $z^t \in \Gamma(\Omega)$ for all $t \geq 0$ then $L^{-1}(0)$ is Lyapunov stable. If, in addition, the RFDE system is strictly $\mathcal{M}$-passive or strictly $\mathcal{M}$-anti-passive then $\mathcal{A}$ is asymptotically stable and globally asymptotically stable if $\mathcal{Y} = \mathcal{M}$.*

Proof: Given $\epsilon > 0$ and $\zeta \in A$, choose $r \in (0, \epsilon]$ such that

$$B_r(A) = \{z \in C([-T, 0], \mathbb{R}^q) : \min_{\xi \in A} \|\xi - z\| \leq r\} \subset \mathcal{Y}.$$

Let

$$Q_r(A, \zeta) = \{z \in \Gamma(B_r(\zeta)) : \min_{\xi \in A} \|\xi - z\| = r\}.$$

If $Q_r(A, \zeta)$ is empty then the theorem is proven, so suppose $Q_r(A, \zeta)$ is non-empty. Let

$$\alpha = \min_{z \in Q_r(A, \zeta)} L(z),$$

let $\beta \in (0, \alpha)$, and let

$$\Omega_{r,\beta}(\zeta) = \{z \in \Gamma(B_r(\zeta)) : L(z) \leq \beta\}.$$

The set $\Omega_{r,\beta}(\zeta)$ must be in the interior of $B_r(A)$ because $\beta < \alpha$. If $z^0 \in \Omega_{r,\beta}(\zeta)$ then $z^t \in \Omega_{r,\beta}(\zeta)$ for all $t \geq 0$, for otherwise there exists $\tau > 0$ such that z^τ is in $Q_r(A, \zeta)$, implying $L(z^\tau) \geq \alpha$, which contradicts $\dot{L} \leq 0$. Now, let δ satisfy $B_\delta(\zeta) \subset \Omega_{r,\beta}(\zeta)$ so that

$$z^0 \in B_\delta(\zeta) \Rightarrow z^t \in \Omega_{r,\beta}(\zeta) \subset B_r(A) \subset B_\epsilon(A),$$

as required. Next, assume the system is strictly $\mathcal{M}$-passive or strictly $\mathcal{M}$-anti-passive. Arguments given above indicate that for any $a > 0$ we can choose $b > 0$ such that $\Omega_{r,b}(\zeta) \subset B_a(A)$, so it is sufficient to show that $L(z^t) \to 0$ as $t \to \infty$. Since $L(z^t)$ is monotonically decreasing and bounded from below by 0, it must have a limit $c \geq 0$ as $t \to \infty$. We reason by contradiction that $c = 0$. Suppose $c > 0$. Continuity of L implies that there exists $d > 0$ such that $\Omega_{r,\beta}(\zeta) - \Omega_{r,c}(\zeta)$ lies outside of $B_d(A)$, so that $L(z^t) \to c > 0$ implies that z^t is outside of $B_d(A)$ for all $t \geq 0$. Let

$$E_{r,d}(\zeta) = \{z \in \Gamma(B_r(\zeta)) : \min_{\xi \in A} \|\xi - z\| \in [d, r]\},$$

and let

$$-\gamma = \max_{z \in E_{r,d}(\zeta)} \dot{L}(z) < 0.$$

It follows that

$$L(z^t) \leq L(z^0) - \gamma t,$$

which contradicts $c > 0$ since the right-hand side must eventually become negative. For global asymptotic stability suppose $c = L(z^0)$ then there exists $r > 0$ such that $\Omega_{0,c}(z^0) \subset B_r(A)$ and the rest of the proof is similar to that given for asymptotic stability. ■

The boundedness requirements of Theorem 5.9.1 are not necessary, but are met by the examples considered in this paper. The function L must satisfy additional conditions if one wishes to provide a passivity definition that leads to stability without additional boundedness assumptions. We can also give an analog of Theorem 5.3.2.

Consider the positive-feedback interconnection of two systems that are, respectively, $\mathcal{M}_1$-anti-passive with storage function L_1 and $\mathcal{M}_2$-passive with storage function L_2.

$$e_1 = -u_1 + h_2(z_2, e_2)$$

$$e_2 = u_2 + h_1(z_1, e_1).$$

In order to extend $\mathcal{M}$-passivity to the interconnection in a meaningful way, we must identify a non-empty set $\mathcal{M}_{12} \subset \mathcal{M}_1 \times \mathcal{M}_2$ so that, for the overall system, $0 \in \mathcal{U}_{\mathcal{M}_{12}}(v)$ for all $v \in \mathcal{M}_{12}$.

Theorem 5.9.2 *The positive-feedback interconnection of two RFDE systems that are, respectively, $\mathcal{M}_1$-anti-passive with storage function L_1 and $\mathcal{M}_2$-passive with storage function L_2, is $\mathcal{M}_{12}$-passive with storage function $L_{12}(z_1, z_2) = L_1(z_1) + L_2(z_2)$.*

The proof is again just a straight application of the classical passivity theorem. For the examples we consider, stability results follow once we show compactness of $\mathcal{M}_{12}$ and apply Theorem 5.9.1.

If the two systems exhibit, respectively, output strict $\mathcal{M}$-anti-passivity, and output strict $\mathcal{M}$-passivity, then passivity follows even for delayed interconnections. In particular, suppose that the two systems are interconnected as

$$e_1(t) = -u_1(t) + h_2(z_2^t(-T_2), e_2^t(-T_2))$$
$$e_2(t) = u_2(t) + h_1(z_1^t(-T_1), e_1^t(-T_1)),$$

which we refer to as the (T_1, T_2)-interconnection of the two systems, where $T_1, T_2 \geq 0$ and without loss of generality $T_1 + T_2 \leq T$. Some authors write these equations as

$$e_1(t) = -u_1(t) + y_2(t - T_2)$$
$$e_2(t) = u_2(t) + y_1(t - T_1),$$

to emphasize that each systems input is the sum of the exogenous input u_i and the lagged output of the other system. We will follow this latter convention to conserve notation. The reader should be advised however that (T_1, T_2)-interconnection will in general require us to redefine T so that $z_i^t(-T_i)$ is meaningful for each i. We once again assume these equations have unique solutions for all (z_1, z_2, u_1, u_2). The (T_1, T_2)-interconnection does not preserve passivity in general. However, the following result, a slightly modified version of Theorem 2 in [99], provides for passivity of the (T_1, T_2)-interconnection given that the two systems exhibit their appropriate forms of output strict passivity.

Theorem 5.9.3 *The (T_1, T_2)-interconnection of two RFDE systems that are, respectively, output strictly $\mathcal{M}_1$-anti-passive with storage function L_1 and output strictly $\mathcal{M}_2$-passive with storage function L_2, is $\mathcal{M}_{12}$-passive.*

Proof: We have

$$
\begin{aligned}
&\dot{L}_1(z_1^t) + \dot{L}_2(z_2^t)\\
\leq& - e_1(t)'y_1(t) + e_2(t)'y_2(t) - \|y_1(t)\|^2 - \|y_2(t)\|^2\\
=& - (-u_1(t) + y_2(t-T_2))'\, y_1(t) - \|y_1(t)\|^2 + \; (u_2(t) + y_1(t-T_1)))'\, y_2(t) - \|y_2(t)\|^2\\
=&u(t)'y(t) - y_2(t-T_2)'y_1(t) + y_1(t-T_1)'y_2(t) - \; \|y_1(t)\|^2 - \|y_2(t)\|^2\\
\leq&u(t)'y(t) + \frac{1}{2}\left(\|y_1(t)\|^2 + \|y_2(t-T_2)\|^2\right) + \frac{1}{2}\left(\|y_2(t)\|^2 + \|y_1(t-T_1)\|^2\right) - \|y_1(t)\|^2 - \|y_2(t)\|^2\\
=&u(t)'y(t) - \frac{1}{2}\left(\|y_2(t)\|^2 - \|y_2(t-T_2)\|^2\right) - \frac{1}{2}\left(\|y_1(t)\|^2 - \|y_1(t-T_1)\|^2\right)\\
\leq&u(t)'y(t) - \frac{1}{2}\frac{d}{dt}\left(\int_{t-T_2}^{t} \|y_2(s)\|^2 ds\right) - \frac{1}{2}\frac{d}{dt}\left(\int_{t-T_1}^{t} \|y_1(s)\|^2 ds\right),
\end{aligned}
$$

where the second inequality follows from the polarization identity and the third inequality follows from Liebniz's rule. We then have

$$\dot{L}_1(z_1^t) + \dot{L}_2(z_2^t) + \dot{L}_{\mathtt{channel}} \leq u(t)'y(t),$$

where

$$L_{\mathtt{channel}} = \frac{1}{2}\left(\int_{t-T_2}^{t} \|y_2(s)\|^2 ds + \int_{t-T_1}^{t} \|y_1(s)\|^2 ds\right).$$

Given $u(t), t \geq 0$ and a solution $(z_1^t, z_2^t), t \geq 0$ and any $\tau \geq 0$ we have that for any $h \geq 0$

$$
\begin{aligned}
&L_1(z_1^{t+h}) + L_2(z_2^{t+h}) - L_1(z_1^t) - L_2(z_2^t)\\
\leq& \int_t^{t+h} u(s)'y(s) ds,
\end{aligned}
$$

since $L_{\mathtt{channel}} \geq 0$. This fact relies on the fundamental theorem of calculus for Dini derivatives. What is required is that the the $L_i, i = 1, 2$ are continuous with finite Dini derivatives that are integrable in the sense of Riemann, Lebesgue, or Denjoy-Perron, see [100]. The above expression will be familiar to many as the integral form of passivity. Substituting $u = 0$ gives a monotonically non-increasing Lyapunov function from which stability results follow. In order to complete the proof of the theorem we must recover the differential form of passivity. Recalling that $L_{12} = L_1 + L_2$ and $z^t = (z_1^t, z_2^t)$

$$\begin{aligned}
\dot{L}_{12}(z^t) &= \limsup_{h\to 0^+} \frac{1}{h}\left(L_{12}(z^{t+h}) - L_{12}(z^t)\right) \\
&\leq \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} u(s)'y(s)ds \\
&= \limsup_{h\to 0^+} u(t + c(h))'y(t + c(h)) = u(t)'y(t),
\end{aligned}$$

where the exisence of $c(h) \in (0, h)$ satisfying the last equality follows from the mean value theorem. ■

The next section introduces a class of learning dynamics formulated as RFDE's.

5.9.3 T-contrarian dynamics

In our earlier examples of dynamic learning, players' substitution of augmented payoffs for nominal payoffs was described within the game subsystem. This was for mathematical convenience— players implemented dynamic learning schemes by employing traditional static learning schemes "as if" their incentives were described by the augmented payoffs. Here, we will find it more convenient to describe the payoff augmentation in the dynamics subsystem directly. In particular, suppose that players perceive advantages in avoiding strategies that have seen net increase in usage during the most immediate T seconds. This is consistent with each agent maintaining a contrarian disposition with respect to the other agents in her population. These percieved incentives are incorporated into the decision-making framework via payoff augmentation

$$\hat{\pi} = \pi - (x(t) - x(t - T)),$$

with the players then reacting to the augmented payoffs. Formally, given a nominal evolutionary dynamic of the usual form

$$\dot{\pi} = u$$
$$\dot{x} = V_{\pi}(x)$$
$$y = \dot{x} = V_{\pi}(x)$$

we define the *T-contrarian subsystem induced by V* as

$$\dot{\hat{\pi}}(t) = u(t) + (V_{\hat{\pi}^t(0)}(x^t(0)) - V_{\hat{\pi}^t(-T)}(x^t(-T)))$$
$$\dot{x}(t) = V_{\hat{\pi}^t(0)}(x^t(0))$$
$$y(t) = \dot{x}(t) = V_{\hat{\pi}^t(0)}(x^t(0)),$$

where $(x^t, \hat{\pi}^t) \in \bar{C}([-T, 0], \mathbb{R}^n \times X)$.

The T-contrarian augmentation preserves passivity.

Theorem 5.9.4 *If V is a (strictly) $\mathbb{R}^n \times X$-passive dynamics subsystem with storage function L_V having a zero level set containing only equilibrium points, then the T-contrarian subsystem induced by V is a (strictly) $\bar{C}([-T, 0], \mathbb{R}^n \times X)$-passive dynamic subsytem with storage function*

$$L_V(x^t(0), \hat{\pi}^t(0)) + \frac{1}{2}\int_{-T}^{0} \|V_{\hat{\pi}^t(s)}(x^t(s))\|^2 ds,$$

having zero level set

$$\{(\pi^t, x^t) : (\pi^t(\theta), x^t(\theta)) = (\bar{\pi}, \bar{x}) \in L_V^{-1}(0) \quad \forall \theta \in [-T, 0]\}.$$

Proof: The storage function L_V satisfies

$$\dot{L}_V(\pi(t), x(t)) \leq (u(t) + y(t - T) - y(t))' y(t).$$

Application of the polarization identity gives

$$\begin{aligned}
\dot{L}_V &= \frac{1}{2}\left(\|y(t-T)\|^2 + \|y(t)\|^2 - \|y(t-T) - y(t)\|^2\right) + u(t)'y(t) - \|y(t)\|^2 \\
&\le \frac{1}{2}\left(\|y(t-T)\|^2 - \|y(t)\|^2\right) + u(t)'y(t) \\
&= u(t)'y(t) - \frac{1}{2}\frac{d}{dt}\int_{t-T}^{t} \|y(s)\|^2 ds \\
&\Rightarrow \dot{L}_V + \dot{L}_{\mathtt{channel}} \le u(t)'y(t),
\end{aligned}$$

where

$$L_{\mathtt{channel}} = \frac{1}{2}\int_{-T}^{0} \|V_{\hat{\pi}^t(s)}(x^t(s))\|^2 ds$$

and the last equality follows from Liebniz's rule. The functional $L_{\mathtt{channel}}$ is clearly non-negative and equal to zero on the set described at the end of the theorem because $L_V^{-1}(0)$ contains only equilibrium points so that $V_{\hat{\pi}^t(s)}(x^t(s)) = 0$ for $s \in [-T, 0]$ whenever $\hat{\pi}^t$, x^t take only constant values in $L_V^{-1}(0)$. ■

The proof of Theorem 5.9.4 was again similar to [99]. Inspection of the zero level set indicates that the infinite-dimensional system induced by T-contrarian tendencies continues to be passive in the appropriate sense with respect to the original equilibrium set, with the points in that set extended to constant-valued functions. Theorem 5.9.4 is conservative in that T-contrarian augmentation preserves passivity even when the form of the learning dynamic is more general. For instance, the theorem goes through even if the dynamic utilizes additional auxiliary states beyond π and x. Furthermore, RFDE learning dynamics can also be accommodated, the simplest example of which is dynamics arrived at by repeated T-contrarian augmentation.

Theorem 5.9.4 says that, from the perspective of passivity, T-contrarian tendencies are without consequence. Guarantees that apply to passive dynamics, such as global convergence to equlibrium in game subsystems induced by stable games, will generally apply to their T-contrarian counterparts. In particular, consider Theorem 5.9.1, the analog of theorem 5.3.1 for RFDE systems. To establish that T-contrarian augmentation of evolutionary dynamics satisfy the conditions of Theorem 5.9.1 (and hence are stable), we must show that for any bounded set $\Omega \subset \bar{C}([-T,0],\mathbb{R}^n \times X)$ there exists a compact set set $\Gamma(\Omega)$ such that $z^0 \in \Omega$ implies $z^t \in \Gamma(\Omega)$ for all $t \geq 0$. Given Ω, let $S(\Omega) \subset \bar{C}([-T,0],\mathbb{R}^n \times X)$ be the states visited by solutions originating in Ω. Boundedness of $S(\Omega)$ clearly follows from the fact that $x^t(\theta) \in X, \theta \in [-T,0]$ and

$$\begin{aligned}
\hat{\pi}(t) &= \hat{\pi}(0) + \int_0^t \left(V_{\hat{\pi}^s(0)}(x^s(0)) - V_{\hat{\pi}^s(-T)}(x^s(-T))\right) ds \\
&= \hat{\pi}(0) + \int_0^t \left(\dot{x}(0) - \dot{x}(t-T)\right) \\
&= \hat{\pi}(0) + \left(x^t(0) - x^t(-T)\right) - \left(x^0(0) - x^0(-T)\right).
\end{aligned}$$

Of course, bounded subsets of $\bar{C}([-T,0],\mathbb{R}^n \times X)$ are not necessarily contained in compact subsets of $\bar{C}([-T,0],\mathbb{R}^n \times X)$[6]. However, we find that $cl(S(\Omega))$ is indeed compact. To see this, we make use of the following result (simplified slightly for our purposes), which is classical.

Theorem 5.9.5 *(Arzela-Ascoli) If a subset of $\bar{C}([-T,0],\mathbb{R}^n \times X)$ is equicontinuous and bounded then its closure is compact.*

A subset of $\bar{C}([-T,0],\mathbb{R}^n \times X)$ is *equicontinuous* if given $\theta \in [-T,0]$ and $\epsilon > 0$ there exists a neighbourhood U of θ such that $\|g(\theta) - g(\phi)\| < \epsilon$ for all $\phi \in U$ and all g in the subset. Equicontinuity of $S(\Omega)$ follows from boundedness of $\dot{x}$ and $\dot{\hat{\pi}}$, which implies that $\Gamma(\Omega) = cl(S(\Omega))$ suffices.

Intuitively, T-contrarian tendencies are sensible in settings where players prefer to be isolated in their strategy choices, such as congestion games. All else being equal, strategies that have seen a net increase in usage over the most recent T seconds are less attractive than strategies that have seen less of an increase or a decrease over the same period. Such behavior is especially advantageous when payoffs are delayed. This is because the disutility caused by the recent uptick in usage is not yet reflected in the observed payoffs. In a class of games including certain congestion games with increasing costs we find that these intuitions translate into a concrete guarantee, described below.

[6]Recall non-compactness of the unit ball.

Theorem 5.9.6 *The* $(0, T)$*interconnection of*

- *a* C^1*game subsystem with all eigenvalues of* $DF(x)$ *in* $[-2, -1]$ *for all* $x \in X$
- *a* T*-contrarian dynamic subsystem induced by a strictly* $X \times \mathbb{R}^n$*-passive evolutionary dynamic* V*,*

is $\mathcal{M}_{\mathcal{G},D}$*-passive, where the set* $\mathcal{M}_{\mathcal{G},D}$ *is comprised of states* $(x^t_{\mathcal{G}}, \pi^t, x^t_{\mathcal{D}}, \tilde{\pi}^t)$ *satisfying*

- $x^t_{\mathcal{G}} = x^t_{\mathcal{D}}$,
- $\pi^t(\theta) = F(x^t_{\mathcal{G}}(\theta)) \quad \forall \theta \in [-T, 0]$
- $\tilde{\pi}^t(0) = \pi^t(0) + x^t_{\mathcal{D}}(0) + x^t_{\mathcal{D}}(-T)$.

Consequently the system admits the globally asymptotically stable set

$$\{\tilde{\pi}^t = \pi^t = \overline{\pi} \in \mathbb{R}^n, x^t_{\mathcal{D}} = \overline{x} \in X \ s.t. \ (\overline{x}, \overline{\pi}) \in L_V^{-1}(0)\}.$$

Since the statement of Theorem 5.9.6 is a bit cumbersome, we first attempt to clarify. Theorem 5.9.6 essentially says that if players augment their payoffs with T-contrarian tendencies and then employ a standard passive dynamic on the augmented payoffs, then they will converge to equilibrium in the specified games, even if they experience delays in receiving the true payoffs. What is required is that the lookback period T match the time delay T that the payoffs are subject to. Additionally, the restriction on the eigenvalues of $DF(x)$ ensures that payoffs are not either too sensitive or too insensitive to strategy changes. The class of games covered by the theorem includes certain congestion games with increasing costs.

By substituting into the equations for the dynamic and game subsystems according to the relations specified by $\mathcal{M}_{\mathcal{G},D}$ it can be verified that the overall system is simply

$$\dot{x}(t) = V_{\hat{\pi}}(x(t)),$$

as claimed. Furthermore, the equilibrium set is just the natural extension of the equilibrium set of V itself. Thus with separable, acute EPT dynamics for instance, players achieve global congvergence to precisely the set $NE(F)$, while with logit they would converge to $PE(F)$. All the infinite-dimensional states converge to constant functions that correspond to Nash or perturbed equilibria of the underlying game exactly.

Proof of Theorem 5.9.6: Our technique is to first show that this interconnection can be transformed into a $(0,T)$-interconnection of two modified subsystems, each exhibiting their respective forms of output strict passivity, and then apply Theorem 5.9.3.

Step 1: Transform the systems. Consider the game subsystem given by

$$
\begin{aligned}
\dot{x}(t) &= u(t) \\
\dot{\pi}(t) &= DF(x(t))u(t) \\
y(t) &= \dot{\pi}(t) + \dot{x}(t) = (DF(x(t)) + I)u(t)
\end{aligned}
$$

along with the dynamics subsystem given by,

$$
\begin{aligned}
\dot{\hat{\pi}}(t) &= u(t) - V_{\hat{\pi}(t)}(x(t)) \\
\dot{x}(t) &= V_{\hat{\pi}(t)}(x(t)) \\
y(t) &= \dot{x}(t) = V_{\hat{\pi}(t)}(x(t)).
\end{aligned}
$$

The $(0,T)$-interconnection of these two finite dimensional systems is equivalent to the interconnection described in the theorem. We will use the subscripts $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{D}}$ to distinguish the variables in respectively, the modified game and dynamics subsystems.

Step 2: Show $-u'_{\tilde{\mathcal{G}}}y_{\tilde{\mathcal{G}}} - \|y_{\tilde{\mathcal{G}}}\|^2 \geq 0$. Note that this inequality implies that the modified game subsystem is strictly output X-anti-passive with storage function 0. Restriction to inputs that keep $x_{\tilde{\mathcal{G}}}$ in the X is understood. To verify the inequality let $\tilde{M}(x_{\tilde{\mathcal{G}}}) = DF(x_{\tilde{\mathcal{G}}}) + I$ and observe that

$$
-u'_{\tilde{\mathcal{G}}}y_{\tilde{\mathcal{G}}} - \|y_{\tilde{\mathcal{G}}}\|^2 = -u'_{\tilde{\mathcal{G}}}(\tilde{M}(x_{\tilde{\mathcal{G}}}) + \tilde{M}^2(x_{\tilde{\mathcal{G}}}))u_{\tilde{\mathcal{G}}},
$$

where symmetry of $\tilde{M}(x_{\tilde{\mathcal{G}}})$ follows from the eigenvalue assumptions. Next, let

$$
Q^{-1}(x_{\tilde{\mathcal{G}}})\Lambda(x_{\tilde{\mathcal{G}}})Q(x_{\tilde{\mathcal{G}}}) = \tilde{M}(x_{\tilde{\mathcal{G}}}),
$$

where for each $x_{\tilde{\mathcal{G}}} \in X$, $\Lambda(x_{\tilde{\mathcal{G}}})$ is a diagonal matrix with elements on the main diagonal equal to the eigenvalues of $\tilde{M}(x_{\tilde{\mathcal{G}}})$. It follows that

$$\begin{aligned} -u'_{\tilde{\mathcal{G}}} y_{\tilde{\mathcal{G}}} - \|y_{\tilde{\mathcal{G}}}\|^2 &= -u'_{\tilde{\mathcal{G}}} Q^{-1}(x_{\tilde{\mathcal{G}}})(\Lambda(x_{\tilde{\mathcal{G}}}) + \Lambda^2(x_{\tilde{\mathcal{G}}}))Q(x_{\tilde{\mathcal{G}}})u_{\tilde{\mathcal{G}}} \\ &\geq 0. \end{aligned}$$

The inequality follows from negative semidefiniteness of $\Lambda(x_{\tilde{\mathcal{G}}}) + \Lambda^2(x_{\tilde{\mathcal{G}}})$, which follows from the elements on the main diagonal of $\Lambda(x_{\tilde{\mathcal{G}}})$ being in $[-1, 0]$ for all $x_{\tilde{\mathcal{G}}} \in X$.

Step 3: Show $u'_{\tilde{\mathcal{D}}} y_{\tilde{\mathcal{D}}} - \|y_{\tilde{\mathcal{D}}}\|^2 \geq L_V(x_{\tilde{\mathcal{D}}}, \tilde{\pi})$. This is *output strict $X \times \mathbb{R}^n$-passivity*. To verify the inequality simply note that

$$(u_{\tilde{\mathcal{D}}} - y_{\tilde{\mathcal{D}}})' y_{\tilde{\mathcal{D}}} \geq L_V(x_{\tilde{\mathcal{D}}})$$

follows directly from $X \times \mathbb{R}^n$-passivity of the subsystem induced by the evolutionary dynamic V, since the modified game subsystem is just that system with its input set to $u_{\tilde{\mathcal{D}}} - y_{\tilde{\mathcal{D}}}$.

Step 4: Apply Theorem 5.9.3 and verify the set $\mathcal{M}_{\mathcal{G},D}$. This step is almost immediate and is thus left to the reader, completing the proof. ■

A more general version of Theorem 5.9.6 can be proven that allows T-contrarian augmentations of non-traditional passive dynamics to achieve the same guarantees. Furthermore, the requirement on the eigenvalues of $DF(x)$ can be relaxed so that they need only reside in $[-2\alpha, \alpha]$ for any $\alpha > 0$ so long as we add complementary scaling to the T-contrarian augmentation. Put another way, the most negative eigenvalue has magnitude less or equal to twice the magnitude of the least negative eigenvalue.

The requirement that the lookback period T in the T-contrarian augmentation match the time delay T that payoff measurement is subject to is not essential. The two time horizons only need to be sufficiently close, which follows from inherent robustness properties of Lyapunov stability results. However, we do not provide a rigorous argument to this end here.

5.10 Discussion

The fact that so many learning dynamics are so well-behaved in the stable games can be explained by the passivity properties of the relevant subsystems. We also showed that convergence properties of other learning paradigms can also be described using passivity. One issue in particular remains open— the status of the replicator dynamics. The Lyapunov function typically employed for replicator dynamics in stable games is the KullbackLeibler divergence between the current social state and the social state at equilibrium. Our framework does not allow the storage function of the learning subsystem to exploit properties of an underlying game structure.

CHAPTER 6

CONCLUSIONS

The preceding three chapters described the contributions of this thesis to the area of distributed learning. Some of these results were aimed at more specific applications while others were more general in scope. In this final conclusion, we adopt a bird's eye view and highlight two central research themes that are especially well-motivated in light of the results described above.

6.1 Convergence rates and metastability

Much of the research in distributed learning has emphasized the prospect of convergence to equilibrium without paying much attention to the amount of time required. This has changed in recent years with a number of new results providing both good and bad news concerning convergence rates [68], [101], [102], [103], [104]. This list not exhaustive and the study of convergence rates in distributed learning has picked up considerably very recently with many new results appearing. These papers for the most part bound the time to convergence for deterministic dynamics or the mixing time for stochastic dynamics. In engineering settings, the objective is to identify distributed algorithms that reach an acceptable outcome sufficiently quickly. In natural and social sciences where the agenda is essentially descriptive— how long does the relevant dynamic take to reach the predicted outcome? In this case, when convergence is extremely slow, the validity of the supposed prediction is called into question. Instead we should be interested in describing the transient medium-run behavior of the model. These middle-ground outcomes are called metastable. For instance, in Section 4.7 we saw that the stochastically stable states were only observed after a very lengthy mixing time, but the simulations were nonetheless highly stable and predictable in the linguistic community structures that they exhibited in the initial phase. Unfortunately, we were not able to support this observation with rigorous analysis. At this point, rigorous characterizations have been limited to very simple cases [72], [105], [106]. The identification of procedures for characterizing metastable states in more general setting seems a paramount concern for distributed learning.

6.2 The relevance of equilibrium

In this thesis we have assumed throughout that the appropriate way to arrive at predictions is by modeling the decision-makers iterative processes directly. Readers unacquainted with the game theory literature more generally will be surprised to learn that this approach is somewhat unconventional. Equilibrium analysis has historically been the dominant endeavour without much reference to what sort of disequilibrium process is required to reach said equilibrium. This is not necessarily problematic, except for a particular consequence we now describe.

Research in distributed learning and particularly game theoretic learning has been especially preoccupied with achieving convergence to equilibrium and in particular Nash equilibrium outcomes. It is possible that this preoccupation stems from historical considerations. Even in this thesis perhaps undue attention was paid to justifying the prediction of Nash equilibrium play. Of course, everywhere that equilibrium analysis has appeared, refinements motivated by dynamic considerations have been offered up. For instance, stochastic stability in the evolution of social conventions [14] and efficiency of equilibria, [107] and evolutionarily stable states (ESS) in biology [108]. Nonetheless, the focus on Nash equilibria remains, provoking some authors to point out more and more drawbacks to this disposition, such as inefficiency relative to dynamic outcomes [109].

Nash equilibria is certainly relevant in a setting like stable games where convergence to equilibrium is so well-established. However, the analysis proceeds in perhaps the wrong direction. That is, researchers look for settings where convergence to Nash is achieved by a wide range of learning dynamics. Researchers in distributed learning may instead begin by classifying dynamics by their satisfaction of certain axioms. Then the appropriate set of dynamics can be identified for a particular strategic setting and the chips can be allowed to fall where they may. If metastable, chaotic, or cycling behavior is predicted, then so long as the dynamics under study are justifiable, then these outcomes should not be written off a priori.

Several classes of learning dynamics have been identified. Some prominent examples include positively correlated dynamics, no regret dynamics, and now passive dynamics. These classes are typically understood in terms of the equilibrium set they eventually reach, i.e. correlated equilibrium with no regret dynamics in all finite games, and Nash equilibria for positively correlated and passive dynamics in, respectively, potential games and stable games. Since non-equilibrium outcomes are not controversial, research perhaps ought to think about what sort of exogenous criteria would lead us to prefer one of these classes over another for a particular model. If this is not possible, then perhaps new classes should be considered that are better suited to this exercise.

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